Separation of variables and explicit solutions of some integrable systems with cubic integrals

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Algebraic Integrable systems

Let a Hamiltonian system in $\mathbb{R}^N = \{x_1, \ldots, x_N\}$ be integrable with m independent *polynomial* integrals $H_1(x), \ldots, H_m(x)$, generic invariant manifolds $\mathcal{I}_h = \{H_1 = h_1, \ldots H_m = h_m\} = \mathbb{T}^g$, $g = N - m$.

It is also *Algebraic completely Integrable* (Adler & van Moerbeke) if

• The *complex* invariant manifolds $\mathcal{I}_h^{\mathbb{C}}$ are open subsets of *Abelian varieties* $\mathcal{A}_h$:

$$\mathcal{I}_h^{\mathbb{C}} \subset \mathcal{A}_h = \text{complex tori } \mathbb{C}^g/\Lambda \subset \mathbb{P}^N \ : \quad \mathcal{I}_h^{\mathbb{C}} = \mathcal{A}_h \setminus \mathcal{D}$$

• The complex flow on $\mathcal{A}_h$ can be *linearlized*.

Main property: All the solutions are *meromorphic* functions of the complex time t.

Examples: The integrable cases of the Kirchoff equaitions (Clebsch, Steklov), the Kovalevskaya top, the Frahm–Manakov top on $so(n)$, the Neumann system on S^n, ...

Algebraic Integrable systems (II)

A particular case: The g-dim. Abelian variety $\mathcal{A}_h$ is the *Jacobian variety* of an algebraic curve Γ of genus g.

The case $g=2$: The system can be reduced to quadratures

$$\frac{d\lambda_1}{\sqrt{R_5(\lambda_1)}} + \frac{d\lambda_2}{\sqrt{R_5(\lambda_2)}} = du_1, \qquad \frac{\lambda_1 d\lambda_1}{\sqrt{R_5(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{R_5(\lambda_2)}} = du_2,$$

$$\Gamma = \{\mu^2 = P_5(\lambda)\}, \quad u_1, u_2 \text{ are linear functions of time } t \in \mathbb{C}$$

generating the *Abel map* $\{\Gamma \times \Gamma\} \to \mathrm{Jac}(\Gamma) = \mathbb{C}^2/\Lambda$

$$\int_{\lambda_0}^{\lambda_1} \omega_1 + \int_{\lambda_0}^{\lambda_2} \omega_1 = u_1, \qquad \int_{\lambda_0}^{\lambda_1} \omega_2 + \int_{\lambda_0}^{\lambda_1} \omega_2 = u_2 . \qquad \omega_1 = \frac{d\lambda}{\sqrt{R_5(\lambda)}}, \ \omega_2 = \frac{\lambda\, d\lambda}{\sqrt{R_5(\lambda)}}$$

It can be inverted by means of theta-functions of u_1, u_2 associated with Γ.
For a genus g curve, to be invertible, the Abel map $\Gamma^{(g)} \to \mathrm{Jac}(\Gamma)$ must contain *all* the g holomorphic differentials on Γ

Examples of non-algebraic integrable systems

- **An elementary example**: a system with one degree of freedom with

$$H = \frac{1}{2}p^2 + \frac{1}{2}V_n(q) = h, \qquad V_n(q) \ = \text{a polynom. of degree } n$$

Then

$$\frac{dq}{dt} = \sqrt{2h - V_n(q)} \Longrightarrow \int_{q_0}^{q} \frac{dx}{\sqrt{2h - V_n(q)}} = t - t_0.$$

- $n = 3, 4 \quad \Longrightarrow \quad q(t) =$ an elliptic function of $t \in \mathbb{C}$
- $n \geq 5$:

The solution $q(t)$ lives on an infine covering of $\{t\} = \mathbb{C}$ ramified over an infinite number of (branch) points, whose projections on $\mathbb{C}$ form a dense set.

$n = 5$: The Puisseau expansions near the critical poles:

$$q(t) = \frac{1}{(t - t_0)^{2/3}}(q^{(0)} + q^{(1)}(t - t_0)^{1/3} + q^{(2)}(t - t_0)^{2/3} + \cdots),$$

Examples of non-algebraic integrable systems (II)

- A generalized Henon–Heiles system on $\mathbb{R}^2$ with the integrable Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V^{(5)}, \qquad V^{(5)} = y^5 + y^3 x^2 + \frac{3}{16} y x^4,$$

which is separable (takes a Stäckel form) in the parabolic coordinates μ_1, μ_2:

$$x^2 = -4\mu_1\mu_2, \quad y = \mu_1 + \mu_2.$$

Once the constants of motion are fixed, the system can be reduced to quadratures

$$\int_{\mu_0}^{\mu_1} \frac{d\mu}{\sqrt{R(\mu)}} + \int_{\mu_0}^{\mu_2} \frac{d\mu}{\sqrt{R(\mu)}} = \text{const},$$

$$\int_{\mu_0}^{\mu_1} \frac{\mu\, d\mu}{\sqrt{R(\mu)}} + \int_{\mu_0}^{\mu_2} \frac{\mu\, d\mu}{\sqrt{R(\mu)}} = 2t + \text{const},$$

$$R(\mu) = \mu(c\mu - d - \mu^6)$$

which contain 2 holomorphic differentials on the genus 3 hyperelliptic curve $\Gamma = \{w^2 = R(\mu)\}$.

- Until recently, no known examples of such systems realted to non-hyperelliptic curves...

Certain integrable systems on T^*S^2 with a cubic integral

The unit sphere $S^2 = \{\langle\gamma,\gamma\rangle = 1\}$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$,
$T^*S^2 = \{\langle\gamma,\gamma\rangle = 1, \langle\gamma, p\rangle = 0\}$. The $so(3)$-momentum $J = \gamma \times p$.
The geodesic flow on S^2 is described by the classical *Kirchhoff equations*

$$\dot{J} = J \times \frac{\partial H}{\partial J} + \gamma \times \frac{\partial H}{\partial \gamma}, \quad \dot{\gamma} = \gamma \times \frac{\partial H}{\partial J}$$

- The Goryachev case:

$$H_1 = J_1^2 + J_2^2 + \frac{4}{3}J_3^2 + \frac{a\gamma_1 + b}{\gamma_3^{2/3}}, \qquad a, b \text{ being arbitrary constants}$$

$$H_2 = -\frac{2}{3}J_3(J_1^2 + J_2^2 + \frac{8}{9}J_3^2 + \frac{a\gamma_1 + b}{\gamma_3^{2/3}}) + a x_3^{1/3} J_1.$$

- The Dullin–Matveev case:

$$H_1 = J_1^2 + J_2^2 + \left(1 + \frac{x_3}{x_3 + c} - \frac{x_3^2 - 1}{4(x_3 + c)^2}\right) J_3^2 + \frac{a\gamma_1}{\sqrt{x_3 + c}} + \frac{b}{x_3 + c},$$

$$H_2 = H_1 J_3 - J_3^2 - \frac{a}{2\sqrt{x_3 + c}}(x_1 J_3 + 2(x_3 + c)J_1).$$

- They are particular cases of a family found recently by G. Vallent.

Separation of variables for the *Goryachev* case

The canonical coordinates and momenta on T^*S^2

$$u = \gamma_3, \quad p_u = \frac{J_1\gamma_2 - J_2\gamma_1}{\gamma_1^2 + \gamma_2^2}, \quad \phi = \arctan(\gamma_1/\gamma_2), \quad p_\phi = -J_3.$$

Following [Vershilov & Tsiganov], separating variables q_1, q_2 are the roots of the polynomial

$$A(\lambda) = \lambda^2 + u^{1/3}\left(\frac{up_\phi}{1-u^2} - \jmath p_u\right)\lambda - \frac{\jmath a e^{\jmath\phi}}{\sqrt{1-u^2}}.$$

The Darboux coordinates q_i, p_i on $T^*\,S^2$

$$\{q_i, p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0,$$

$$p_\phi = \jmath\, q_1 q_2 \frac{p_2 - p_1}{q_1 - q_2}, \quad u = \left[-\frac{2}{3}\jmath \frac{q_1 p_1 - q_2 p_2}{q_1 - q_2}\right]^{3/2}, \qquad \jmath = \sqrt{-1},$$

$$u^{1/3}\left(\frac{up_\phi}{1-u^2} - \jmath p_u\right) = -q_1 - q_2, \quad \frac{\jmath a e^{\jmath\phi}}{\sqrt{1-u^2}} = -q_1 q_2,$$

in which the Hamiltonians take a *Stäckel* form.

The original variables in terms of the separating ones

Let $\lambda_1 = q_1, \lambda_2 = q_2,\ \mu_1 = 2/3\,\jmath q_1 p_1,\ \mu_2 = 2/3\,\jmath q_2 p_2$, then

$$\gamma_3^2 = -\left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^3, \quad \gamma_3^{2/3} = -\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2},$$

$$J_3 = -p_\phi = -\frac{3}{2}\,\frac{\lambda_1\mu_2-\lambda_2\mu_1}{\lambda_1-\lambda_2},$$

$$\gamma_2+\jmath\gamma_1 = \frac{2\jmath}{a}\,\lambda_1\lambda_2\left(1+\left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^3\right), \quad \gamma_2-\jmath\gamma_1 = \frac{a}{2\jmath\,\lambda_1\lambda_2},$$

$$J_1+\jmath J_2 = -\frac{a}{2}\,\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}\left(-\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^{-1/2},$$

$$J_1-\jmath J_2 = \frac{\imath}{\gamma_2-\imath\gamma_1}\left(\imath(J_1+\imath J_2)(\gamma_2+\imath\gamma_1)-2J_3\gamma_3\right) = \cdots$$

The quadratures

Let t_1, t_2 denote the time of the flows on T^*S^2 with the Hamiltonians H_1, H_2.

$$\frac{d\lambda_1}{d\,t_1} = \frac{k}{\mu_1 - \mu_2}\frac{\partial F(\lambda_1,\mu_1)}{\partial\mu_1} \quad \frac{d\lambda_2}{d\,t_1} = \frac{k}{\mu_2 - \mu_1}\frac{\partial F(\lambda_2,\mu_2)}{\partial\mu_2},$$
$$\frac{d\lambda_1}{d\,t_2} = \frac{k\mu_2}{\mu_1 - \mu_2}\frac{\partial F(\lambda_1,\mu_1)}{\partial\mu_1}, \quad \frac{d\lambda_2}{d\,t_2} = \frac{k\mu_1}{\mu_2 - \mu_1}\frac{\partial F(\lambda_2,\mu_2)}{\partial\mu_2},$$

where

$F(\lambda,\mu) = \lambda^4 - b\lambda^2 + (\mu^3 - h_1\mu + h_2)\lambda + a^2/4 = 0,$ (the Goryachev system)

$F(\lambda,\mu) = \mu^3\lambda + c\lambda^2\mu(c\lambda + 2\mu) - (h_1 + \lambda^2)\lambda\mu + h_2\lambda + \frac{a^2}{4} - b\lambda^2 = 0$ (D-M)

$S = \{F(\lambda,\mu) = 0\}$ is an algebraic curve of genus 3.
The equations are equivalent to

$$\frac{d\lambda_1}{\partial F(\lambda_1,\mu_1)/\partial\mu_1} + \frac{d\lambda_2}{\partial F(\lambda_2,\mu_2)/\partial\mu_2} = k\,dt_2$$
$$\frac{\mu_1 d\lambda_2}{\partial F(\lambda_1,\mu_1)/\partial\mu_1} + \frac{\mu_2 d\lambda_2}{\partial F(\lambda_2,\mu_2)/\partial\mu_2} = k\,dt_1.$$

The quadratures II

By making the birational change

$$\lambda = \frac{1}{x}\sqrt{\frac{a}{2}}, \quad \mu = \frac{y}{x}\sqrt{\frac{a}{2}}$$

the genus 3 curve S is transformed to the *canonical trigonal form*

$$G \equiv y^3 + p(x)y + q(x) = 0,$$

$$p(x) = -2\frac{h_1}{a}x^2, \quad q(x) = x^4 + \frac{2\sqrt{2}h_2}{a^{3/2}}x^3 - 2\frac{b}{a}x^2 + 1$$

having one triple infinite point ∞ and 3 holomorhic differentials

$$\Omega_1 = \frac{\mathrm{d}x}{\partial G/\partial y}, \quad \Omega_2 = \frac{x\,\mathrm{d}x}{\partial G/\partial y}, \quad \Omega_3 = \frac{y\,\mathrm{d}x}{\partial G/\partial y}, \qquad \frac{\partial G}{\partial y} = 3y^2 - \frac{2h_1}{a}x^2,$$

whereas the quadratures take the form

$$\frac{x_1\,\mathrm{d}x_1}{\partial G(x_1,y_1)/\partial y_1} + \frac{x_2\mathrm{d}x_2}{\partial G(x_2,y_2)/\partial y_2} = k\,\mathrm{d}t_2$$
$$\frac{y_1\mathrm{d}x_1}{\partial G(x_1,y_1)/\partial y_1} + \frac{y_2\,dx_2}{\partial G(x_2,y_2)/\partial y_2} = k\,\mathrm{d}t_1.$$

The quadratures and the full Abel map

That is,

$$\int_{\infty}^{P_1} \begin{pmatrix}\Omega_2\\ \Omega_3\end{pmatrix} + \int_{\infty}^{P_2} \begin{pmatrix}\Omega_2\\ \Omega_3\end{pmatrix} = \begin{pmatrix} k\,t_2 + \text{const}\\ k\,t_1 + \text{const}\end{pmatrix}, \quad P_i = (x_i, y_i) \in S\,,$$

- **The full Abel map** $S^{(3)} \to \text{Jac}(S) = \mathbb{C}^3/\Lambda$

$$\int_{\infty}^{P_1} \mathbf{\Omega} + \int_{\infty}^{P_2} \mathbf{\Omega} + \int_{\infty}^{P_3} \mathbf{\Omega} = \mathbf{u} = \begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix}, \quad \mathbf{\Omega} = \begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix}$$

Can be inverted by means of the theta or *sigma*-function of the trigonal curve S

$$\sigma(\mathbf{u}) = \theta(\mathbf{u} - \mathbf{K}) \exp\left(\frac{1}{2}\mathbf{u}^T \varkappa\, \mathbf{u} + \text{a linear function of } \mathbf{u}\right)$$

Its expansion near the origin

$$\sigma = u_1 - u_2^2 u_3 + \frac{1}{20} u_3^5 + \frac{\mu_2}{6} u_2^2 u_3^3 - \frac{3\mu_2}{504} u_3^7 - \frac{\mu_3}{2} u_2^3 u_2^3 + \text{higher order terms.}$$

Inversion of the full Abel map

Introduce Kleinian multi-index functions

$$\wp_{i,j}(\mathbf{u}) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(\mathbf{u}), \qquad i,j = 1,2,3,$$

$$\wp_{i,j,k}(\mathbf{u}) = -\frac{\partial^2}{\partial u_i \partial u_j \partial u_k} \ln \sigma(\mathbf{u}), \qquad i,j,k = 1,2,3.$$

Theorem (V. Enolskii, D. Leykin)

The inversion of

$$\int_\infty^{P_1} \mathbf{\Omega} + \int_\infty^{P_2} \mathbf{\Omega} + \int_\infty^{P_3} \mathbf{\Omega} = \mathbf{u}, \quad P_i = (x_i, y_i) \in S$$

is given by the 3 solutions (x_i, y_i) of the system

$$\wp_{33}(\mathbf{u})y + \wp_{23}(\mathbf{u})x + \wp_{13}(\mathbf{u}) = x^2,$$
$$(\wp_{23}(\mathbf{u}) - \wp_{333}(\mathbf{u}))y + (\wp_{22}(\mathbf{u}) - \wp_{233}(\mathbf{u}))x + \wp_{12}(\mathbf{u}) - \wp_{133}(\mathbf{u}) = 2xy,$$

For example:

$$x_1 + x_2 + x_3 = 3\wp_{23}(\mathbf{u}) - \wp_{333}(\mathbf{u}).$$

The Wirtinger strata $W^{(0)} = \{0\} \in W^{(1)} \subset W^{(2)} \subset \mathrm{Jac}(S)$

$$W^{(1)}: \quad \mathbf{u} \in \mathrm{Jac}(S), \quad \mathbf{u} = \int_{\infty}^{P} \boldsymbol{\Omega} \quad \forall P \in S$$
$$W^{(2)}: \quad \mathbf{u} \in \mathrm{Jac}(S), \quad \mathbf{u} = \int_{\infty}^{P_1} \boldsymbol{\Omega} + \int_{\infty}^{P_2} \boldsymbol{\Omega} \quad \forall P_1, P_2 \in S\,.$$

Let $\sigma_\alpha = \partial\sigma(\mathbf{u})/\partial u_\alpha$.

Theorem (V. Buchstaber, V. Enolskii, D. Leykin)

The strata $W^{(0)} \in W^{(1)} \subset W^{(2)}$ are given by the conditions

$$\begin{aligned} W^{(0)}: &\quad \sigma(\mathbf{u}) = \sigma_3(\mathbf{u}) = \sigma_2(\mathbf{u}) = 0, \\ W^{(1)}: &\quad \{\mathbf{u} |\ \sigma(\mathbf{u}) = 0,\ \sigma_3(\mathbf{u}) = 0\}, \\ W^{(2)}: &\quad \{\mathbf{u} |\ \sigma(\mathbf{u}) = 0\} \end{aligned}$$

Conclusion: The invariant manifold of the system on T^*S^2 is the 2-dim. stratum $W^{(2)} \subset \mathrm{Jac}(S)$.

Some formal solutions

The quadratures for the system on T^*S^2

$$\int_\infty^{P_1} \mathbf{\Omega} + \int_\infty^{P_2} \mathbf{\Omega} = \begin{pmatrix} u_1 \\ u_2 = kt_2 + \text{const} \\ u_3 = kt_1 + \text{const} \end{pmatrix}$$

$\{\mathbf{u} \in W^{(2)} \iff \sigma(u_1, u_2, u_3) = 0\} \implies u_1$ is transc. function of u_2, u_3.

The condition $\wp_{33}(\mathbf{u})y + \wp_{23}(\mathbf{u})x + \wp_{13}(\mathbf{u}) = x^2$ gives

$$\gamma_3^{2/3} = \frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2} = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} = \frac{x_1 x_2 - \wp_{13}(\mathbf{u})}{\wp_{33}(\mathbf{u})},$$

$$J_3 = \frac{\lambda_2 \mu_1 - \lambda_1 \mu_2}{\lambda_1 - \lambda_2} = \sqrt{\frac{a}{2}} \frac{y_1 - y_2}{x_2 - x_1} = \sqrt{\frac{a}{2}} \frac{x_1 + x_2 - \wp_{23}(\mathbf{u})}{\wp_{33}(\mathbf{u})},$$

then

$$\gamma_3^{2/3} = \left.\frac{\sigma_1(\mathbf{u})}{\sigma_3(\mathbf{u})}\right|_{\sigma(\mathbf{u})=0}, \quad J_3 = \sqrt{\frac{a}{2}} \left.\frac{\sigma_2(\mathbf{u})}{\sigma_3(\mathbf{u})}\right|_{\sigma(\mathbf{u})=0}.$$

The singularity analysis

Choose a point $\mathbf{u}_0 \subset W^{(2)}$. The expansion of $\sigma(\mathbf{u}_0 + \delta u_i)$

$$\sigma_1(\mathbf{u}_0)\delta u_1 + \sigma_2(\mathbf{u}_0)\delta u_2 + \sigma_3(\mathbf{u}_0)\delta u_3 + \sum_{1 \le i,j \le 3} \frac{\sigma_{i,j}(\mathbf{u}_0)}{2}\delta u_i \delta u_j + \cdots = 0$$

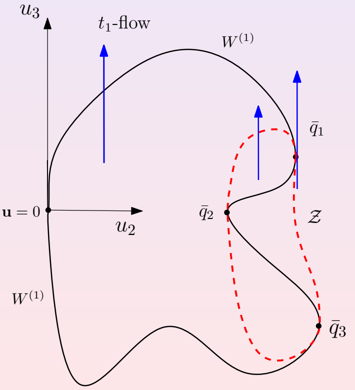
The coordinate u_1 is *not* a locally meromorphic function of u_2, u_3 along the 1-dimensional analytic subvariety

$$\mathcal{Z} = \{\sigma(\mathbf{u}) = 0,\ \sigma_1(\mathbf{u}) = 0\} \subset W^{(2)},$$

which has 3 common points with $W^{(1)}$:

$$\mathcal{Z} \cap W^{(1)} = \{\bar{q}_1, \bar{q}_2, \bar{q}_3\}, \quad \bar{q}_\alpha = \int_\infty^{q_\alpha} \boldsymbol{\Omega},$$

$q_\alpha = (0, y_\alpha)$, a point on S over $x = 0$.
The variables γ, J of the system have poles along $W^{(1)} \subset W^{(2)}$.



Projection of stratum $W^{(2)}$ onto (u_2, u_3)-plane

The local pole (Painlevé) analysis

- The principal ballance (depends on 3 free parameters)

$$\gamma_3 = O\left(\frac{1}{(t-t_0)^{3/2}}\right), \quad J_3 = O\left(\frac{1}{t-t_0}\right),$$
$$\gamma_2 + \imath\gamma_1 = O(1), \qquad \gamma_2 - \imath\gamma_1 = O\left(\frac{1}{(t-t_0)^3}\right),$$
$$J_1 + \imath J_2 = O\left(\frac{1}{(t-t_0)^{5/2}}\right), \quad J_1 - \imath J_2 = O\left((t-t_0)^{1/2}\right).$$

- The secondary ballances (depend on 2 free parameters)

$$\gamma_3 = O(1), \quad J_3 = O\left(1/(t-t_0)\right),$$
$$\gamma_2 + \imath\gamma_1 = O(t-t_0), \quad \gamma_2 - \imath\gamma_1 = O\left(1/(t-t_0)\right),$$
$$J_1 + \imath J_2 = O\left(1/(t-t_0)\right), \quad J_1 - \imath J_2 = O(t-t_0).$$

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