Superintegrable surface metrics admitting one linear and one cubic integral

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Problem

Surface metrics as Hamiltonians

- $(\dim M = 2)$, $g_M = g_{ij}(x)dx^i dx^j$ (Riemannian metric) $\Rightarrow$
- $H_g := \frac{1}{2} g^{ij}(x) p_i p_j$ on T^*M with $\omega_{\mathrm{can}} = dp_i \wedge dx^i$
- **Hamiltonian flow** $\Longleftrightarrow$ **Geodesic flow.**
- **Physical/Mechanical meaning:** Kinetic energy.
- **Maupertuis' principle:** $H = H_g + V(x)$ (Potential) $\Rightarrow$ $\widetilde{H}_g := \frac{H_g}{E - V(x)}$ $\Rightarrow$ Reduction to case without $V(x)$.

Superintegrable surface metrics:

- **A 1st integral:** (X^{2n},ω) (symp. mfld), $H \in C^\infty(X)$ Hamiltonian, $F \in C^\infty(X)$ with $\{H,F\}=0$ (Poisson bracket).
- $H \in C^\infty(X^{2n},\omega)$ is **integrable** $\Leftrightarrow \exists\ n$ integrals $F_1,\dots,F_n$ + non-degeneracy (usually $F_n \equiv H$)
- $H \in C^\infty(X^{2n},\omega)$ is **superintegrable** $\Leftrightarrow \exists\ n+1$ integrals $F_1,\dots,F_{n+1}$ (usually $F_{n+1} \equiv H$)
- **A polynomial integral:** $(X,\omega,H) = (T^*M, \omega_{\mathrm{can}}, H_g)$ and F is polynomial in momenta ($\Leftrightarrow$ velocities): $F = \sum_I F_I(x)p^I$ (where $I=(i_1,\dots,i_d)$ and $p^I = p_{i_1}\cdot\ldots\cdot p_{i_d}$), $d := \mathrm{deg}(F)$.
- $\mathcal{F}^d(H) = \mathcal{F}^d(g)$: polynomial integrals of degree d for $H = \frac{1}{2}g^{ij}(x)p_ip_j$
- **A linear integral** $L = L(x)^i p_i \Leftrightarrow$ **Killing vector field** $v = v^i\partial_{x^i}$
- $H_g = \frac{1}{2}g^{ij}(x)p_ip_j$ is **a quadratic integral**
- **A polynomially superintegrable surface metric:** $X = T^*M$, $\dim M = 2$, $H = \frac{1}{2}g^{ij}(x)p_ip_j$, $\exists$ two polynomial integrals F_1, F_2 for H: $\{H,F_1\} \equiv \{H,F_2\} \equiv 0$.

Previously known cases

- **Two linear integrals** L_1, L_2**:** $\Rightarrow$ g has **constant curvature**, $\mathcal{F}^1(g) = \mathrm{Lie}(\mathrm{Iso}(M, g))$, $\mathcal{F}^d(g) = \{$polynomials in $\mathcal{F}^1(g)\}$
- **Linear + Quadratic integrals** $L + F$**:** [Darboux+Koenigs] $\Rightarrow \exists$ one more quad.int. F_2, $\mathcal{F}^2(g) = \mathbb{R}\langle L^2, H, F, F_2\rangle$ (if g is *not* of const curvature)
  such g and H_g are **Darboux superintegral** metics
- **3 quadratic integrals** F_1, F_2, F_3**:** [Darboux+Koenigs] $\Rightarrow$ g is of constant curvature or Darboux-superintegrable
- **2 quadratic integrals** F_1, F_2**:** [Koenigs]: a complete(?) description
- Next case $\exists$ **linear+cubic** L, F
  Partial results [Rañada,Gravel,Marquette,Winternitz] for the case $H = H_g + V(x)$ where $g = dx^2 + dy^2$ is flat. They show that after Maupertuis' transform $F = L{\cdot}Q$ with *quadratic* integral Q. $\Rightarrow$ Reduction to Darboux-case.

Main result.

Theorem 1

Let (M,g) be a Riemann surface s.th. $H=\frac{1}{2}g^{ij}p_ip_j$ admits (independent) linear and cubic integrals $L+F$. Then $\exists$ coordinates (x,y) s.th. $L=\partial_y$ and $g=\frac{1}{h_x^2}(dx^2+dy^2)$ where $h=h(x)$ satisfies one of the eqns (where $h_x:=\frac{dh(x)}{dx}$)

$$
\begin{aligned}
&(i) && h_x\big(A_0h_x^2+\mu^2A_0h(x)^2-A_1h(x)+A_2\big) = \Big(A_3\frac{\sin(\mu x)}{\mu}+A_4\cos(\mu x)\Big)\\
&(ii) && h_x\big(A_0h_x^2-\mu^2A_0h(x)^2-A_1h(x)+A_2\big) = \Big(A_3\frac{\sinh(\mu x)}{\mu}+A_4\cosh(\mu x)\Big)\\
&(iii) && h_x\big(A_0h_x^2-A_1h(x)+A_2\big) = \big(A_3x+A_4\big)
\end{aligned}
$$

with some real constants $A_0,\dots,A_4$ and $\mu>0$ in cases (i,ii). In all three cases $\mathcal{F}^3(g)=\langle L^3, L\cdot H, F_1, F_2\rangle$ (4-dimensional).

The explicit formulas for F_1, F_2 are of the form:

(i) $F_1(x,y)=\cosh(\mu y)\cdot f_{i,1}$, $F_2(x,y)=\sinh(\mu y)\cdot f_{i,2}$, (trig. $\sin,\cos$ in eqn)

(ii) $F_1(x,y)=\cos(\mu y)\cdot f_{ii,1}$, $F_2(x,y)=\sin(\mu y)\cdot f_{ii,2}$, (hyperb. $\sinh,\cosh$ in eqn)

with some *polynomials* $f_{i,ii;1,2}$ in $h(x), h_x, h_{xx}, h_{xxx}$ (and in p_x, p_y). Case (iii) is similar.

Remarks. • The curvature of $g = \frac{(dx^2+dy^2)}{h_x^2}$ is $R = h_{xxx}h_x - h_{xx}^2$.

- The equations (i–iii) are defined for *complex* $x, A_0, \ldots, A_4, \mu$, solution $h(x)$ depends complex-analytically on them and on initial value $h(x_0)$, and real-analytically if all are real. A solution $h(x)$ of a *complex* eqn is real only in the cases (i–iii).
- We exchange cases (i)↔(ii) making substitution $\mu \leftrightarrow \mathbf{i}\mu$ ($\mathbf{i} := \sqrt{-1}$) and obtain case (iii), including integrals F_1, F_2, letting $\mu \longrightarrow 0$ in cases (i) and (ii).
- Case $A_0 = 0 \Leftrightarrow$ Darboux-superintegrable (or const-curvature)
- Constant curvature case $\Leftrightarrow$ $h(x)$ is
    - (a) a polynimial in x of $\deg \leqslant 2$, or
    - (b) $h(x) = c\sin(\mu x + \varphi_0)$, or
    - (c) $h(x) = c_+\exp(\mu x) + c_-\exp(-\mu x)$
- g is *not* const-curvature ⇒ the eqn on $h(x)$ is unique (also in complex case) ⇒ the metrics $g = \frac{(dx^2+dy^2)}{h_x^2}$ are new.
- **Big gap phenomenon.** $\operatorname{maxdim} \mathcal{F}^3(H) = 10$ $(R_g = \mathrm{const})$ $\operatorname{submaxdim} \mathcal{F}^3(H) = 4$

Second main result.

Theorem 2 (Global solution on the sphere S^2.)

Assume that parameters of the eqn

$$h_x\big(A_0(h_x^2 - h(x)^2) - A_1 h(x) + A_2\big) \;=\; \big(A_3\sinh(x) + A_4\cosh(x)\big) \qquad \text{(ii)}$$

satisfy $A_0 > 0$, $A_4 > |A_3|$ (and $\mu = 1$). Let $h(x)$ be a unique local solution of the eqn (ii) such that $h_x(x_0) > 0$. Then the solution $h(x)$ extends to the whole line $x \in \mathbb{R}$ and the metric $g = \frac{(dx^2+dy^2)}{h_x^2}$ extends to a real-analytic metric on the Rimann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ with the complex coordinate $z = e^{x+\mathbf{i}y}$. Moreover, the cubic integrals F_1, F_2 above also extend real-analytically on the whole S^2.

Remarks.

- $\theta := \arctan(e^{-x})$ and $\varphi := y$ are spherical coordinates on S^2.
- Condition $A_4 > |A_3|$ means that r.h.s. of (ii) is > 0. In this case the *algebraic* eqn $\lambda(\lambda^2 - a) = A > 0$ has unique *positive* root $\eta = \eta(a, A)$.

Previous results on global completely integrable metrics.

- Many classical examples (Lagrange, Euler) of integrable metrics on S^2 with linear or quadratic first integral.
- Kovalevskaya top (1889): Metric on S^2 with $\deg F = 4$.
- Goryachev(-Chaplygin) top: (1916) $\deg F = 3$
- Classification of linear or quadratic integrable metrics on S^2 and T^2 (Kolokoltsov, Kiyohara, Bobenko-Nehoroshev, Matveev,...) (1984-...)
- Selivanova,(1999), Dullin-Matveev,(2004), Tsyganov,(2005), generalised by Valent,(2010): metric on S^2 admitting cubic F,
- Kiyohara,(2001), metric on S^2 with F of any degree $d \geqslant 3$
- Kiyohara,(1991), If g on S^2 admits quadratic F_1, F_2 $(\dim \mathcal{F}^2(g) \geqslant 3) \Rightarrow R_g \equiv \mathrm{const}$.
- M-Sh,(2011): The first one which is polynomially superintegrable.

Steps of the proof of Theorem 1 (local case).

1. Choose coordinates s.th. $g = \lambda(x)(dx^2 + dy^2)$ and $L = p_y$.
$\dim \mathcal{F}^d(g) < \infty$ (Kruglikov). $L \in \mathcal{F}^1(g)$, $F \in \mathcal{F}^d(g) \Rightarrow \{L, F\} \in \mathcal{F}^d(g)$
and by assumption $\mathcal{F}^3(g)$ contains $L^3, L{\cdot}H, F$
$\Rightarrow \mathcal{F}^3(g)$ contains Evector w.r.t. L with Evalue $\mu \neq 0 \in \mathbb{C}$,
or $\exists\ F \in \mathcal{F}^3(g)$ s.th. $\{L, F\} = A_3 L^3 + A_1 L{\cdot}H$
$\Leftrightarrow$ $F = e^{\mu y}\tilde{F}(x)$ or $F = \tilde{F}(x) + y(A_3 L^3 + A_1 L{\cdot}H)$ with
$\tilde{F}(x) = \sum_{i=0}^{3} a_i(x) p_x^{3-i} p_y^i$
Substitution in $\{H, F\} = 0$ gives $\lambda(x) = \frac{1}{h_x^2}$, $g = \frac{dx^2+dy^2}{h_x^2}$ and
$a_0(x) = A_0 h_x^3$, $a_1(x) = (-\mu A_0 h(x) + \frac{A_1}{\mu}) h_x^2$, and so on, where
A_0, A_1, A_2 are integration constant (the same as in eqns (i–iii)).
$h(x)$ must satisfy certain ODE $\mathcal{E}_3(h)$ of *order 3*.

2. It appears that $\mathcal{E}_3(h) = (\partial_x^2 + \mu^2)\mathcal{E}_1(h)$ for some ODE $\mathcal{E}_1(h)$ of order 1. $\Rightarrow$ Eqn on $h(x)$ is $\mathcal{E}_1(h) = A_3 \frac{\sin(\mu x)}{\mu} + A_4 \cos(\mu x)$ or …
This gives Thm1 in cases (i,ii).

3. $F = L{\cdot}Q \Leftrightarrow a_0 \equiv 0 \Leftrightarrow A_0 = 0$ (Darboux-superint. case)

4. Uniqueness Theorem. If $R = h_{xxx}h_x - h_{xx}^2 \neq \mathrm{const}$ and $h(x)$ satisfies one of (i–iii) $\Rightarrow$ parameters $A_0, \ldots, A_4$ are unique up to const and μ up to sign.

5. Corollary: if h_x is real-valued $\Rightarrow A_0, \ldots, A_0$ are real (up to common complex factor) and $\mu^2 \in \mathbb{R}$. $\Rightarrow$ Only cases (i–iii) yield genuine Riemannian metric.

6. Corollary: $\dim \mathcal{F}^3(g) \geqslant 4$ in cases (i,ii).

7. $\dim \mathcal{F}^3(g) = 4$. In cases (i,ii): Show that $(\partial_y - \mu)^2 F = 0$ satifying $\{F, H\} = 0$ exists only if $(\partial_y - \mu)F = 0$
In case (iii): Solve $\partial_y^2 F = A_3 L^3 + A_1 L \cdot H$ satifying $\{F, H\} = 0$ and show that $\partial_y^3 F = A_3 L^3 + A_1 L \cdot H$ is unsolvable.
Technique is the same as for Uniqueness Thm.

8. Global solution. $\Leftarrow$ Fine analysis of $h(x)$ for $x \longrightarrow \pm\infty$.
$\Leftrightarrow$ Behaviour of g and F at Northern and Southern pole.

Open Questions

1. Quantisation. Find PDOs $\hat{H}, \hat{F}$ with symbols H, F such that $[\hat{H}, \hat{F}] = 0$. Is it possible for $\hat{H} = \Delta_g$ (Laplace-Beltrami) ?

2. Construct concrete (and steel) model, physical or mechanical dynamical systems realising metrics in Thm1 or Thm2.

3. Pseudo-Riemannian case?

4. Find isometric embedding of global metrics (Thm2) in $(\mathbb{R}^3, g_{\mathrm{st}})$ and find condition when such embeddings exist

5. Higher degrees? (Some progress in case $L + Q$ "linear+quartic").

Thank you for your attention!!!