Polynomial integrals and Rich quasi-linear systems of conservation laws.

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Introduction

- Darboux, Birkhoff: problem of Polynomial additional integrals for natural Hamiltonian systems.
- Poinacare: periodic orbits and splitting of separatrices.
- Kozlov-Denisova-Treschev: The case of finitely many Fourier modes for coefficients.
- New examples with higher degree of the polynomial integrals:
  Bolsinov-Fomenko, Selivanova, Kiyohara, Dullin-Matveev, Tsiganov, Valent

<u>Main idea</u>:

System of Equations on coefficients of the polynomial integral has a very remarkable structure: it is Semi-Hamiltonian system of Hydrodynamic type (in the sense of Dubrovin, Novikov…). Moreover for this class of systems blow up analysis along characteristics can be performed (Lax, Serre…).

Models

- Model (U):Systems with 1,5 degrees of freedoms

$$H = p^2/2 + u(q,t), \quad F(p,q,t) = \frac{1}{n+1}p^{n+1} + u_0 p^n + u_1 p^{n-1} + \cdots + u_n$$

- Model (G):Geodesic flows on two-torus.

  Two types of coordinate systems: conformal and semi-geodesic (Fermi,equi-distant…)

$$ds^2 = \Lambda(dq_1^2 + dq_2^2), \quad ds^2 = g^2(t,x)dt^2 + dx^2$$

B.-A.Mironov: for geodesic flows with polynomial integral

$$F = \sum_{k=0}^{n} \frac{a_k(t,x)}{g^{n-k}} p_1^{n-k} p_2^k, \qquad a_{n-1} \equiv g \text{ and } a_n \equiv 1$$

The semi-geodesic coordinates always exist globally on the covering of the torus

- Model (M): Magnetic geodesic flows with exact magnetic fields on surfaces.

$$\omega = \omega_0 + \pi^* \sigma, \sigma = d\alpha$$

Quasi-linear systems

In all three models one gets systems of quasi-linear PDEs:

- Model(U):

$$U = (u_1, .., u_n)^t$$

$$U_t + A(U)U_q = 0, \quad A(U) = -\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ (n-1)u_1 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2u_{n-2} & 0 & 0 & \cdots & 0 & -1 \\ u_{n-1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

- Model (G) , semi-geodesic coordinates:

$$U = (a_0, \dots, a_{n-2}, a_{n-1})^t, \quad U_t + A(U)U_x = 0$$

$$A(U) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \dots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \dots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \dots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}.$$

- Geometric meaning: It turns out that the eigenvalues of the matrix A in both cases correspond to those momenta of the phase space where invariant torii have vertical tangency

- Model (M). We are not aware of an analog of semi-geodesic coordinates, For conformal metric the equations take the form

$$A(U)U_x + B(U)U_y = 0$$

It turns out that for all three models these equations have remarkable structure: they are Rich or Semi-Hamiltonian systems of Hydrodynamic type. This is not evolution form and requires special analysis.

- Quasi-linear systems of PDEs

$U_t + A(U)U_x = 0$
Strict Hyperbolicity:
all eigenvalues λ_i of $A(U)$are real and distinct.

Riemann invariants, Semi-Hamiltonian or Rich property

- Given $U_t + A(U)U_x = 0 \ , \ U = (u_1, ..., u_n)$

  The system is called diagonalizable if there exist new variables called Riemann invariants, $(r_1, ...r_n) \leftrightarrow (u_1, ..., u_n)$

  such that the system takes the diagonal form:

$$(r_i)_t + \lambda_i(r_1, ..., r_n)(r_i)_x = 0, i = 1, ..., n.$$

- Semi-Hamiltonian or Rich Systems: diagonalizable +condition

(R) $$\partial_{r_k}\left(\frac{\partial_{r_i}\lambda_j}{\lambda_i - \lambda_j}\right) = \partial_{r_i}\left(\frac{\partial_{r_k}\lambda_j}{\lambda_k - \lambda_j}\right), i \neq j \neq k \neq i$$

(R) appeared in B.Dubrovin and S.Novikov and studied by S.Tsarev and independently B.Sevennec and D.Serre; later on studied by many people Gibbons, Kodama, Mokhov, Pavlov, Ferapontov…

- Hamiltonian systems of Hydrodynamic type which are diagonalizable are Semi-Hamiltonian but not necessarily vice versa.

Conservation laws

- Conservation law is a pair of functions of field variables (entropy and flux) satisfying : $g_t + h_x = 0.$
- Rich systems have infinitely many conservation laws

  (parameterized by n arbitrary functions of one variable Tsarev, Sevennec)
- Criterion of Richness:

  Theorem (Sevennec). *Quasi- linear system in Riemann invariants* $(r_i)_t + \lambda_i(r_1, ..., r_n)(r_i)_x = 0, i = 1, ..., n.$

  is Rich iff it can be written in the form of n-conservation laws

  $$(g_i)_x + (h_i)_y = 0, i = 1, ..., n$$

  and the functions $g_i(r_1, ..., r_n),\ i = 1, ..., n$ *are independent.*

Models (U),(G) and (M) are Rich

- Theorem1 (B, B-Mironov)

  In the strictly hyperbolic region the system (U) and (G) are Rich and (M) is Rich in a generalized sense .

  Proof: The idea is to use Sevennec criteria: one needs to construct

  1.Riemann Invariants and

  2. Conservation laws.

  For the model (U):

$$H = p^2/2 + u(q,t),\ F(p,q,t) = \frac{1}{n+1}p^{n+1} + u_0 p^n + u_1 p^{n-1} + \cdots + u_n$$

  Invariance under the Hamiltonian flow $F_t + pF_q - u_q F_p = 0$

  Find the solutions of $F_p = 0 \Rightarrow p = \lambda_i(u_1, ..., u_n)$

  And set $r_i = F(\lambda_i, u_1, ..., u_n) \Rightarrow (r_i)_t + \lambda_i (r_i)_x = 0$

- Conclusion: Eigenvalues of the system are critical points of the integral F

  With respect to the fibre and Critical values of F are Riemann invariants.

Models (U),(G) and (M) are Rich

- Construction of infinitely many conservation laws for the system
  That are constructed with the help of invariant torii of the flow:
  Suppose $p = f(q,t)$ is an invariant torus of the flow.

$$f_t + f f_q + u_q = 0 \Rightarrow f_t + (f^2/2 + u)_q = 0$$

  Define $f(u_1, ..., u_n)$ to be the solution of the equation

$$F(f, u_1, ..., u_n) = c$$

  Every such f is a conservation law.
- Conclusion: invariant torii of the integrable flow give rise to conservation laws of the quasi-linear system.

  Riemann invariants + conservation laws imply that the system is Rich.

  Theorem1 is proved.

Blow up along characteristics

Lax analysis for Rich quasi-linear systems (satisfying (R)):
For a given system in Riemann invariants,

(1) $(r_i)_t + \lambda_i(r_1, ..., r_n)(r_i)_x = 0, i = 1, ..., n.$

Introduce characteristic vector fields $v_i = \partial_t + \lambda_i \partial_x$ and $w_i = (r_i)_x$
Differentiate (1) with respect to x to get

(2) $L_{v_i}(w_i) + w_i^2(\lambda_i)_{r_i} + w_i \sum_{j \neq i}(\lambda_i)_{r_j}(r_j)_x = 0.$

One can easily verify $(r_j)_x = \frac{L_{v_i} r_j}{\lambda_i - \lambda_j}$, so that (2) gets the form:

(3) $L_{v_i}(w_i) + w_i^2(\lambda_i)_{r_i} + w_i \sum_{j \neq i}(\lambda_i)_{r_j} \frac{L_{v_i} r_j}{\lambda_i - \lambda_j} = 0.$

Richness condition (R) : there exists a function G_j, such that $\partial_{r_i} G_j = \frac{(\lambda_j)_{r_i}}{\lambda_i - \lambda_j}$
and so (3) gets the form

(4) $L_{v_i}(w_i) + w_i^2(\lambda_i)_{r_i} - w_i L_{v_i} G_i = 0.$ Multiplying (4) by $\exp(-G_i)$

Blow up

One comes to the Riccati equation along characteristics:

$$L_{v_i}(\exp(-G_i)(w_i)) + (\exp(G_i)(\lambda_i)_{r_i})(\exp(-G_i)w_i)^2 = 0.$$

- Reminder

Given Riccati equation

$$\dot{z} + K(t)z^2 = 0 \Rightarrow z(t) = \frac{z_0}{1+z_0\int_{t_0}^{t} K(s)ds}$$

So if the integral diverges the blow up occurs in a finite time

- Genuine nonlinearity condition:

$\frac{\partial \lambda_i}{\partial r_i}$ does not change sign.

Integrals of degree 3,4

- For the model (U) largest and smallest eigenvalues in Hyperbolic case are always genuinely non-linear.

  Theorem 2 (B) *The case of cubic and quartic integral of motion F for system (U) is always reducible, i.e. there exists an integral of degree 1 or 2 and F can be expressed through them.*
- Q: Is it true for higher degrees of n?
- R:Interestingly enough system (U) is equivalent to so called Benney chain.
- For the model (G) the following results can be proved by this method:

  Let Ω be a domain on the two torus were not all of the eigenvalues of the matrix A(U) are real.

  Theorem 3

Let $n = 3$, then one has the following alternative:
Either metric is flat in the region Ω or F_3 is reducible on Ω , that is it can be written as combination of H and F_1

$$F_3 = k_1 F_1^3 + 2k_2 H F_1$$

for some explicit constants k_1, k_2.

Integrals of degree 3,4

- <u>Corollary4</u>

If the Riemannian metric is given in a conformal way $ds^2 = \Lambda(q_1, q_2)(dq_1^2 + dq_2^2)$, then either metric is flat on Ω or $\Lambda = \Lambda(mq_1 + nq_2)$ on Ω for some reals m, n; If in addition the metric is known to be real analytic on the torus then $\Lambda = \Lambda(mq_1 + nq_2)$ everywhere on the whole torus and the geodesic flow necessarily has a first power integral on the whole torus.

- <u>Theorem5</u>

Let $n = 4$, then the following alternative holds: Either metric is flat on Ω or F_4 is reducible, that is it can be expressed on Ω as

$$F_4 = k_1 F_2^2 + 2k_2 H F_2 + 4k_3 H^2$$

where F_2 is a polynomial of degree 2 which is an integral of the geodesic flow on Ω and k_i are constants.

Integrals of degree 3,4

- <u>Corollary 6</u>

  The conformal factor $\Lambda(q_1, q_2)$ *can be written on* Ω *in the form*

$$\Lambda(q_1, q_2) = f(m_1 q_1 + n_1 q_2) + g(m_2 q_1 + n_2 q_2) \quad with \quad \frac{m_1}{n_1}\frac{m_2}{n_2} = -1.$$

If in addition Λ *is known to be real analytic then* Λ *can be written in such a form for all* q_1, q_2 *on the two torus.*

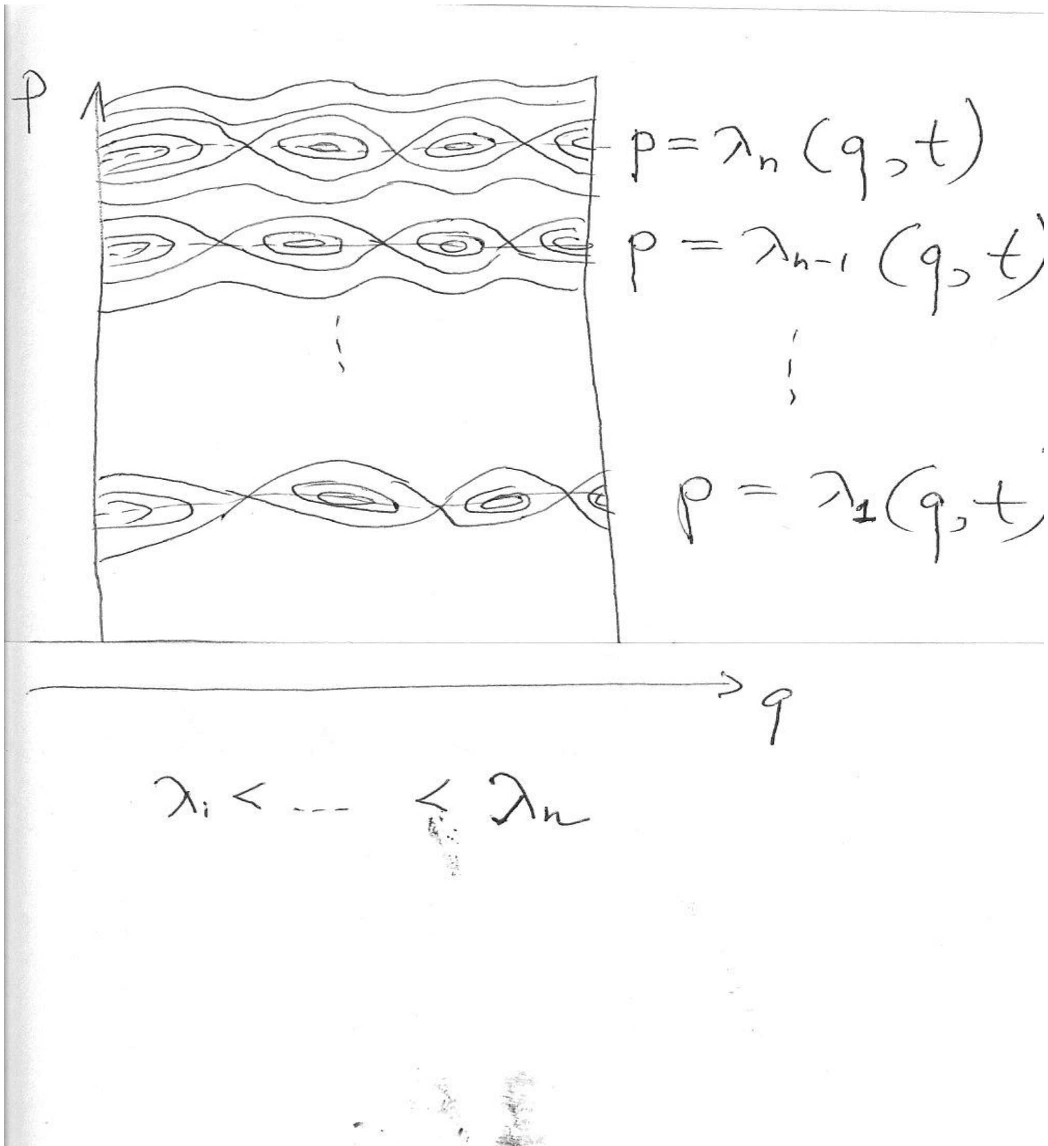
- <u>Conclusion</u>

  These results show that if new integrable examples exist they could be found only within the region of Hyperbolicity of the quasi-linear system.

- Geometric meaning of the Hyperbolic regions.

  Hyperbolicity prescribes a very precise phase portrait of the Hamiltonian flow. Let me draw for the model (U):

Phase portrait



Questions, Remarks

- Q: for the model (U) what about genuine nonlinearity of intermediate eigenvalues?
- Remark: There is an analog of the Richness for quasi-linear systems which are not in evolution form.

  This may help to understand models (G) and (M) further.
- Q:Magnetic geodesic flows (Joint with Andrey Mironov). Search for polynomial integrals for magnetic flows also leads to Rich Systems of PDEs.

  For this problem the case of degree 2 is already not known and not-trivial.
- Q:It is an interesting question if the systems (G) and (M) are Hamiltonian systems of Hydrodynamic type. What is the Poisson bracket?

Papers:

- Bialy M. On Periodic solutions for a reduction of Benney chain
- Bialy M. Integrable geodesic flows on surfaces
- Bialy M, Mironov A. Rich quasi-linear system for integrable geodesic flows on 2-torus
- Bialy M, Mironov A. Cubic and Quartic integrals for geodesic flow on 2-torus via system of Hydrodynamic type
- Bialy M, Mironov A. Integrable magnetic geodesic flows and Rich quasi-linear systems. In preparation.