Differential Galois obstructions to integrability. An overview

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Outline

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Integrability by quadratures

We consider a general system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{v}(\mathbf{x}), \quad \mathbf{x} := (x_1, \ldots, x_n) \in U \in \mathbb{R}^n. \tag{E}$$

We solve the system just using elementary functions (whatever it means) and elementary operations:

1. algebraic operations between functions;
2. compositions of functions (highly non-elementary operation);
3. integration (highly non-elementary operation);
4. inversion of function (highly non-elementary operation);

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Integrability by means of first integrals

Definition

A differentiable function $F : U \to \mathbb{R}$ is a first integral of system (E) iff $L_{\mathbf{v}}(F) = 0$.

Differentiable functions $F_i : U \to \mathbb{R}$, $i = 1, \ldots, k$ are functionally independent on a set $V \subset U$ iff gradients $\nabla F_1(\mathbf{x}), \ldots, \nabla F_k(\mathbf{x})$ are linearly independent at each $\mathbf{x} \in V$.

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First integral as a tensor invariant

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{v}(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0$$

Phase flow generated by vector field $\mathbf{v}(\mathbf{x})$

$$\mathbb{R}^n \supset U \ni \mathbf{x}_0 \mapsto \mathbf{x}(t, \mathbf{x}_0) := \mathbf{\Phi}_t(\mathbf{x}_0)$$

F is a first integral iff $\mathbf{\Phi}_t^\star(F) = F$ for all $t \in \mathbb{R}$ $\quad\Leftrightarrow\quad$ $L_{\mathbf{v}}(F) = 0.$

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Tensor invariants

A tensor field $\mathbf{T}(\mathbf{x})$ with coordinates $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ is an invariant of the system iff $\mathbf{\Phi}_t^\star(\mathbf{T}) = \mathbf{T}$ for all $t \in \mathbb{R}$ $\quad\Leftrightarrow\quad L_{\mathbf{v}}(\mathbf{T}) = \mathbf{0}$.

Theorem (Lie)

If a system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ admits n linearly independent symmetry fields $\mathbf{w}_1 = \mathbf{v}, \ldots, \mathbf{w}_n$, such that

$$[\mathbf{w}_i, \mathbf{w}_j] = \mathbf{0} \quad \textit{for} \quad i, j \in \{1, \ldots, n\},$$

then it is integrable by quadrature.

The Last Jacobi Multiplier

An invariant n-form $\Omega = f(\mathbf{x})\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ of the system is called the Jacobi Last Multiplier of the system.

$$L_{\mathbf{v}}(\Omega)(\mathbf{x}) := \mathrm{div}(f(\mathbf{x})\mathbf{v}(x))\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Theorem (Jacobi)

If a system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ admits Jacobi last multiplier and $n-2$ functionally independent first integrals, then it is integrable by quadrature.

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B integrability

Definition (B-integrability)

A system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ is B-integrable iff it admits $1 \leq k \leq n$ linearly independent symmetry vector fields $\mathbf{w}_1 = \mathbf{v}, \ldots, \mathbf{w}_k$, and $n-k$ functionally independent first integrals $F_1, \ldots, F_{n-k}$ such that

$$[\mathbf{w}_i, \mathbf{w}_j] = \mathbf{0} \quad \text{and} \quad L_{\mathbf{w}_i}(F_l) = 0,$$

for all $i, j \in \{1, \ldots, k\}$ and all $l \in \{1, \ldots, n-k\}$.

Our problem

Almost always

A system is integrable in the Somebody sense if it admits N_{S} tensor invariants (which satisfy certain additional conditions).

Integrability in the Somebody sense implies integrability by quadratures.

Problem

Find necessary conditions for various types of integrability

...method?

Hint

In the frame of the of the differential Galois approach to the integrability.

A solution helps! – Variational equations

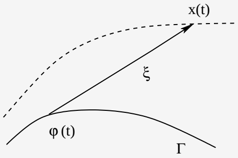
Main Idea

A non-linear system leaves fingerprints of its properties in variational equations.

For a system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \qquad \mathbf{x} = (x_1, \ldots x_n)^T,$$

with known particular solution $\varphi(t)$ the substitution $\mathbf{x} = \varphi(t) + \boldsymbol{\xi}$ and linearization gives



variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi} = A(t)\boldsymbol{\xi}, \qquad A(t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\varphi(t)).$$

Leading terms

Definition

The leading term F° of a holomorphic function F is the lowest order term of an expansion

$$F(\varphi(t)+\boldsymbol{\xi}) = F_m(\boldsymbol{\xi}) + O(\|\boldsymbol{\xi}\|^{m+1}), \qquad F_m \neq 0,$$

i.e., $F^\circ(\boldsymbol{\xi}) := F_m(\boldsymbol{\xi})$. Note that $F^\circ(\boldsymbol{\xi})$ is a homogeneous polynomial with respect to $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ of degree m and its coefficients are polynomials in $\varphi(t)$.

Definition

If F is a meromorphic function, then $F = P/Q$ for certain holomorphic functions P and Q. In this case, the leading term F° of F is defined as $F^\circ = P^\circ/Q^\circ$, where P° and Q° are leading terms of P and Q, respectively. Hence, $F^\circ(\boldsymbol{\xi})$ is a homogeneous rational function of $\boldsymbol{\xi}$.

First integrals of the system and its VEs

Implication

If F is a meromorphic (holomorphic) first integral of the differential system, then its leading term F° is a rational (polynomial) first integral of variational equations. Similarly, if the system possesses $k \geq 2$ functionally independent meromorphic first integrals $F_1, \ldots, F_k$, then, by the Ziglin Lemma, VEs have k functionally independent rational first integrals. Warning: generally they are NOT leading terms of $F_1, \ldots, F_k$!

Question: how to find first integrals of linear systems ?

Outline

Linear equations and Galois theory

Idea

For a linear system of dimension n

$$\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\xi} = A\boldsymbol{\xi}, \qquad A \in \mathbb{M}(n, \mathbb{F}),$$

or a scalar equation of degree n

$$\mathcal{L}y = 0, \quad \partial^n + f_1\partial^{n-1} + \cdots + f_{n-1}\partial + f_n, \quad \partial = \frac{\mathrm{d}}{\mathrm{d}z}, \ f_i \in \mathbb{F}.$$

we ask when we can solve them effectively (by a combination of quadratures, exponential of quadratures and algebraic functions). Old history: Liouville, Picard, Vessiot, Ritt, Kolchin, ...

Question: Is it possible to connect the solvability result for variational equation with the integrability of our nonlinear system?

Monodromy and differential Galois group

For an linear differential equation with coefficients in a differential field $\mathbb{F}$, the Picard-Vessiot extension $\mathbb{L} \supset \mathbb{F}$ contains all its solutions.

Definition

The differential Galois group $\mathcal{G} = \mathcal{G}(\mathbb{L}/\mathbb{F})$ of differential extension $\mathbb{L} \supset \mathbb{F}$ it a subgroup of differential automorphisms of $\mathbb{L}$ which do not move elements of $\mathbb{F}$

Differential Galois group

For an arbitrary solution y and arbitrary $\phi \in \mathcal{G}(\mathbb{L}/\mathbb{F})$

$$0 = \phi(\mathcal{L}y) = \mathcal{L}(\phi y) \Longrightarrow \phi y \text{ is again a solution.}$$

Elements of $\mathcal{G}(\mathbb{L}/\mathbb{F})$ acts as $\mathbb{C}_{\mathbb{F}}$-linear maps on the n-dimensional vector space of solutions.

Differential Galois group

- $\mathcal{G}$ is a linear algebraic group, subgroup of $\mathrm{GL}(n, \mathbb{C}_{\mathbb{F}})$,
- $\mathcal{G}$ has a unique connected component $\mathcal{G}^{\circ}$ which contains the identity.
- $g(f) = f$ for all $g \in \mathcal{G} \iff f \in \mathbb{F}$.
- All solutions of $\mathcal{L}(y) = 0$ are Liouvillian iff $\mathcal{G}^{\circ}$ is solvable.

Correspondence between first integrals of the system and invariants of DGG

Theorem

If system has k functionally independent first integrals which are meromorphic in a connected neighbourhood of a non-equilibrium solution $\varphi(t)$, then the differential Galois group $\mathcal{G}$ of the variational equations along $\varphi(t)$ has k functionally independent rational invariants.

$$\mathbb{C}(\mathbf{x})^{\mathcal{G}} := \{ f \in \mathbb{C}(\mathbf{x}) \mid g \cdot f = f \quad \text{for all} \quad g \in \mathcal{G} \}$$

Fact

The differential Galois group $\mathcal{G}$ of a system of linear equations is a linear algebraic group, so in particular it is also a Lie group.

Passing to Lie algebras

$\mathfrak{g} \subset \mathrm{gl}(n, \mathbb{C})$ – the Lie algebra of $\mathcal{G}$.

With a $Y \in \mathfrak{g}$ we connect a linear vector field:

$$\mathbf{x} \mapsto Y(\mathbf{x}) := Y \cdot \mathbf{x}$$

for $\mathbf{x} \in \mathbb{C}^n$.

Definition

$f \in \mathbb{C}(\mathbf{x})$ is an integral of $\mathfrak{g}$, iff $L_Y(f) = 0$ for all $Y \in \mathfrak{g}$.

$$\mathbb{C}(\mathbf{x})^{\mathfrak{g}} := \{ f \in \mathbb{C}(\mathbf{x}) \mid L_Y(f) = 0 \quad \text{for all} \quad Y \in \mathfrak{g} \}$$

Lemma

If $f_1, \ldots, f_k \in \mathbb{C}(\mathbf{x})^{\mathcal{G}}$ are algebraically independent invariants of an algebraic group $\mathcal{G} \subset \mathrm{GL}(n, \mathbb{C})$, then $f_1, \ldots, f_k \in \mathbb{C}(\mathbf{x})^{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$.

Outline

Liouville Integrability for Hamiltonian systems

$$\frac{\mathrm{d}}{\mathrm{d}t}q_i = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, n$$

$H_i = H_i(\mathbf{q}, \mathbf{p})$, for $i = 1, \ldots, n$ are first integrals,

$$\{H_i, H_j\} := \sum_{k=1}^{n} \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} = 0$$

Definition

Hamiltonian system with n degrees of freedom is integrable in the Liouville sense if it admits n functionally independent commuting first integrals.

Morales-Ramis theorem for Hamiltonian systems

$$\dot{z} = \mathbb{J}H'(z), \quad \mathbb{J} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}, \quad z = [q, p]^T$$

$H(z)$ – a holomorphic function
VE along a particular solution $\varphi(t)$

$$\dot{Y} = \mathbb{J}H''(\varphi(t))Y.$$

The differential Galois group of VEs is a subgroup of $\mathrm{Sp}(2n, \mathbb{C})$.

Hamiltonian systems — Morales-Ramis theorem

Theorem

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of a phase curve Γ. Then the identity component of the differential Galois group of the variational equations along Γ is Abelian.

- Morales Ruiz, J. J., *Differential Galois theory and non-integrability of Hamiltonian systems*, volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.
- Audin, M., *Les systèmes hamiltoniens et leur intégrabilité*, Cours Spécialisés 8, Collection SMF, SMF et EDP Sciences, Paris, 2001.

Outline of the proof

1. Commuting independent first integrals $F_1, \ldots, F_n$ of X_H give rational, commuting and independent first integrals $f_1, \ldots f_n$ of variational variational equations (Ziglin)
2. Thus, $f_1, \ldots f_n \in \mathbb{C}(\mathbf{x}, \mathbf{y})^{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of the differential Galois group of variational equations.
3. Missing point:

Lemma (Key Lemma)

If a Lie algebra $\mathfrak{g} \subset \mathrm{sp}(2n, \mathbb{C})$ admits n independent and commuting first integrals, then it is Abelian.

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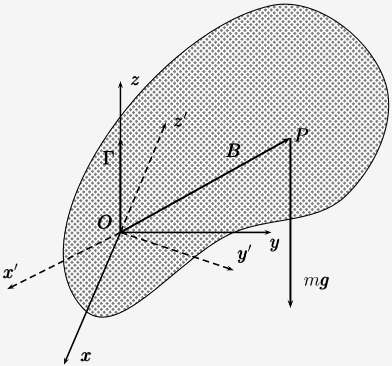
Euler-Poisson equations



$$\dot{\mathbf{M}} + \mathbf{\Omega} \times \mathbf{M} = mg\,\mathbf{\Gamma} \times \mathbf{B},$$
$$\dot{\mathbf{\Gamma}} + \mathbf{\Omega} \times \mathbf{\Gamma} = 0$$

$$H = \frac{1}{2}\langle \mathbf{M}, I^{-1}\mathbf{M}\rangle + \langle \mathbf{\Gamma}, \mathbf{B}\rangle,$$
$$\mathcal{I}_1 = \langle \mathbf{\Gamma}, \mathbf{\Gamma}\rangle$$
$$\mathcal{I}_2 = \langle \mathbf{M}, \mathbf{\Gamma}\rangle$$

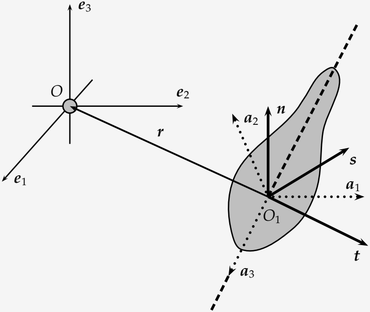
Known integrable cases

- completely symmetric: $I_1 = I_2 = I_3$,
- Euler: $B_1 = B_2 = B_3 = 0$,
- Lagrange: $I_1 = I_2,\ B_1 = B_2 = 0$,
- Kovalevskaya: $I_1 = I_2 = 2I_3,\ B_3 = 0$.

Fundamental question

Do there exist other integrable cases? No!

Satellite in geo-magnetic field



$$\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\Omega} + 3\mathbf{S} \times \mathbf{I}\mathbf{S} + \xi\langle \mathbf{L}, \mathbf{N}\rangle \mathbf{L} \times \mathbf{N},$$
$$\dot{\mathbf{S}} = \mathbf{S} \times (\boldsymbol{\Omega} - \mathbf{N}),$$
$$\dot{\mathbf{N}} = \mathbf{N} \times \boldsymbol{\Omega}.$$

$$H = \frac{1}{2}\langle \mathbf{M}, \mathbf{I}^{-1}\mathbf{M}\rangle - \langle \mathbf{M}, \mathbf{N}\rangle + \frac{3}{2}\langle \mathbf{S}, \mathbf{I}\mathbf{S}\rangle - \frac{1}{2}\xi\langle \mathbf{L}, \mathbf{N}\rangle^2$$
$$H_2 = \langle \mathbf{S}, \mathbf{S}\rangle, \ H_3 = \langle \mathbf{N}, \mathbf{N}\rangle, \ H_4 = \langle \mathbf{N}, \mathbf{S}\rangle, \ H_5 = M_3.$$

If satellite is axially symmetric and $2\xi \neq 3(I_3 - 1)$, then it does not admit an additional meromorphic first integral.

Other applications

- cosmological models: Bianchi, FRW, etc
- restricted two body problems in $\mathbb{S}^2$ and $\mathbb{L}^2$ with Kepler and Hooke interactions.
- generalized two fixed center problem
- Gross-Neveu, spring-pendulum systems,
- generalized Jacobi problem
- a simple proof of non-integrability of the three body problem, ...

Morales-Ruiz, Juan J. and Ramis, Jean-Pierre, Integrability of dynamical systems through differential Galois theory: a practical guide, Contemp. Math., vol. 509, pp. 143–220, 2010.

General results

1. Necessary conditions for non-commutative integrability.
   - A.J. Maciejewski, M. Przybylska, Differential Galois obstructions for non-commutative integrability. Phys. Lett. A **372** (33):5431-5435, (2008)
2. Necessary conditions for super-integrability.
   - A.J. Maciejewski, M. Przybylska, H. Yoshida, Necessary conditions for super-integrability of Hamiltonian systems, Phys. Lett. A **372** (34):5581-5587, (2008)
3. Necessary conditions for partial-integrability.
   - A.J. Maciejewski, M. Przybylska, Partial integrability of Hamiltonian systems with homogeneous potentials. Regul. Chaotic. Dyn. vol. 15(4-5), pp. 551-563, 2010
4. Necessary conditions for B-integrability.
   - N. Ayoul, N, Tien Zung, Galoisian obstructions to non-Hamiltonian integrability, C, R. Acad. Sci., vol. 348, pp. 1323-1326, 2010.

Outline

Darboux Points and Particular Solutions

Assumption

Potential $V \in \mathbb{C}(\mathbf{q})$ is homogeneous and $\deg V = k \in \mathbb{Z}^\star$.

Definition

Darboux point $\mathbf{d} \in \mathbb{C}^n$ is a solution of

$$V'(\mathbf{d}) = \mathbf{d}, \qquad \mathbf{d} \neq \mathbf{0}.$$

Particular solution

$$\mathbf{q}(t) = \varphi(t)\mathbf{d}, \quad \mathbf{p}(t) = \dot{\varphi}(t)\mathbf{d}, \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

Phase curve Γ_ε:

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right)$$

Other Morales-Ramis Theorem

Variational equations

$$\ddot{\mathbf{x}} = -\varphi(t)^{k-2} V''(\mathbf{d})\mathbf{x}.$$

If $V''(\mathbf{d})$ is diagonalisable, then in an appropriate base

$$\ddot{y}_i = -\lambda_i \varphi(t)^{k-2} y_i, \qquad 1 \leq i \leq n, \tag{1}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $V''(\mathbf{d})$. One of these eigenvalues, let us say λ_n is $k-1$.

Theorem

If the Hamiltonian system with homogeneous potential is meromorphically integrable, then each (k, λ_i) belong to the following list:

Morales-Ramis table

case	k	λ
1.	± 2	λ
2.	k	$p + \dfrac{k}{2}p(p-1)$
3.	k	$\dfrac{1}{2}\left(\dfrac{k-1}{k} + p(p+1)k\right)$
4.	3	$-\dfrac{1}{24} + \dfrac{1}{6}(1+3p)^2, \quad -\dfrac{1}{24} + \dfrac{3}{32}(1+4p)^2$ $-\dfrac{1}{24} + \dfrac{3}{50}(1+5p)^2, \quad -\dfrac{1}{24} + \dfrac{6}{25}(1+5p)^2$
5.	4	$-\dfrac{1}{8} + \dfrac{2}{9}(1+3p)^2$

Morales-Ramis table

case	k	λ	
6.	5	$-\frac{9}{40}+\frac{5}{18}(1+3p)^2$,	$-\frac{9}{40}+\frac{2}{5}(1+5p)^2$
7.	-3	$\frac{25}{24}-\frac{1}{6}(1+3p)^2$,	$\frac{25}{24}-\frac{3}{32}(1+4p)^2$
		$\frac{25}{24}-\frac{3}{50}(1+5p)^2$,	$\frac{25}{24}-\frac{6}{25}(1+5p)^2$
8.	-4	$\frac{9}{8}-\frac{2}{9}(1+3p)^2$	
9.	-5	$\frac{49}{40}-\frac{5}{18}(1+3p)^2$,	$\frac{49}{40}-\frac{2}{5}(1+5p)^2$

where p is an integer and λ an arbitrary complex number.

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Weakness of Morales-Ramis theorem in applications

$$V = \frac{1}{3}aq_1^3 + \frac{1}{2}q_1^2 q_2 + \frac{1}{3}cq_2^3.$$

$$\lambda_1 = \frac{1}{c}, \qquad \lambda_{2,3} = \frac{2c-1}{1+a^2 \mp \Delta}, \qquad \Delta = \sqrt{a^2(2+a^2-2c)}$$

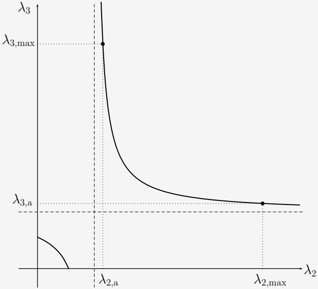
$$\lambda_1, \lambda_2, \lambda_3 \in \left\{ p + \frac{3}{2}p(p-1) \right\} \cup \left\{ \frac{1}{2}\left(\frac{2}{3} + p(p+1)k\right)\right\}$$
$$\cup \left\{ -\frac{1}{24} + \frac{1}{6}(1+3p)^2 \right\} \cup \left\{ -\frac{1}{24} + \frac{3}{32}(1+4p)^2 \right\}$$
$$\cup \left\{ -\frac{1}{24} + \frac{3}{50}(1+5p)^2 \right\} \cup \left\{ -\frac{1}{24} + \frac{3}{50}(2+5p)^2 \right\}$$

$$c = \frac{1}{\lambda_1}, \quad a = \pm\sqrt{\frac{(\lambda_1 + \lambda_1\lambda_i - 2)^2}{2\lambda_1\lambda_i(2-\lambda_1-\lambda_i)}}, \qquad i = 2, 3.$$

Our observation

$$\lambda_1\lambda_2\lambda_3 - (\lambda_1+\lambda_2+\lambda_3)+2=0.$$

one of λ_i, let us say λ_1, must be smaller than 1



There is only a finite number of $(\lambda_1,\lambda_2,\lambda_3)$ satisfying the above relation and the thesis of Morales-Ramis theorem.

$$\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}+\frac{1}{\Lambda_3}=-1, \quad \Lambda_i=\lambda_i-1.$$

Relation for $n = 2$

If

$$V \neq \alpha \left(q_1^2 + q_2^2\right)^{k/2}$$

then there exist at most $k = \deg V$ Darboux points.

Theorem

Assume that there exist $1 \leq l \leq k$ simple Darboux points z_i and the multiplicity of linear factors $(q_2 \pm \mathrm{i} q_1)$ is at most 1, then $\Lambda_i = \Lambda(z_i)$ satisfy

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1$$

Proof:
Apply the global residue theorem for meromorphic 1-form on $\mathbb{CP}^1$ and use the formula $\operatorname{res}_{z^*} \dfrac{h(z)}{g(z)} = \dfrac{h(z^*)}{g'(z^*)}$

Result for $n = 2$ and $k = 3$

$\{\Lambda_1, \Lambda_2, \Lambda_3\}$
$\{-1, -1, 1\}$
$\{-2/3, 4, 4\}$
$\{-7/8, 14, 14\}$
$\{-2/3, 7/3, 14\}$.

Case	Integrable potential
1	$(q_1 + \mathrm{i} q_2)^l (q_1 - \mathrm{i} q_2)^{3-l}, \ l = 0, \ldots, 3$
2	$q_1^3 + c q_2^3/3$
3	$q_1^2 q_2/2 + q_2^3$
4	$q_1^2 q_2/2 + 8 q_2^3/3$
5	$\mathrm{i}\sqrt{3} q_1^3/18 + q_1^2 q_2/2 + q_2^3$

Theorem

Hamiltonian system with homogeneous potential of degree 3 is meromorphically integrable if and only if it belongs to items 1–5.

Results for an arbitrary n

Fact

A generic homogeneous $V \in \mathbb{C}[\mathbf{q}]$ has exactly $D(n,k) = [(k-1)^n - 1]/(k-2)$ Darboux points.

$$\mathcal{D}^{\star}(V) \ni [\mathbf{d}] \longmapsto \mathbf{\Lambda}(\mathbf{d}) = (\Lambda_1(\mathbf{d}), \ldots, \Lambda_{n-1}(\mathbf{d}))$$

where $\lambda_i(\mathbf{d}) := \Lambda_i(\mathbf{d}) + 1$, are the non-trivial eigenvalues of $V''(\mathbf{d})$.
τ_i is the elementary symmetric polynomial of degree i in $(n-1)$ variables.

Results for an arbitrary *n*

Theorem

For a generic homogeneous $V \in \mathbb{C}[\mathbf{q}]$ of degree $k > 2$ we have

$$\sum_{[\mathbf{d}]\in\mathcal{D}^\star(V)} \frac{\tau_1(\mathbf{\Lambda}(\mathbf{d}))^r}{\tau_{n-1}(\mathbf{\Lambda}(\mathbf{d}))} = (-1)^n(n+k-2)^r, \quad 0 \leq r \leq n-1,$$

or, alternatively

$$\sum_{[\mathbf{d}]\in\mathcal{D}^\star(V)} \frac{\tau_r(\mathbf{\Lambda}(\mathbf{d}))}{\tau_{n-1}(\mathbf{\Lambda}(\mathbf{d}))} = (-1)^{n-r-1}\sum_{i=0}^{r}\binom{n-r-1}{r-i}(k-1)^i.$$

Results for an arbitrary n

Theorem

For a generic homogeneous $V \in \mathbb{C}[\mathbf{q}]$ of degree k set of admissible $\{\boldsymbol{\Lambda}(\mathbf{d}) \mid [\mathbf{d}] \in \mathcal{D}^{\star}(V)\} =: \mathcal{I}_{n,k}$ is finite.

$\Downarrow$ +many other things

New integrable potentials for $k = n = 3$

Example for $n = k = 3$

$$\Lambda_1(\mathbf{d}_j), \Lambda_2(\mathbf{d}_j) \in B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6, \quad j = 1, \ldots, 7,$$

$$B_1 = -1 + \frac{1}{2}p(3p-1), \qquad B_2 = -\frac{2}{3} + \frac{3}{2}p(p+1),$$

$$B_3 = -\frac{25}{24} + \frac{1}{6}(1+3p)^2, \quad B_4 = -\frac{25}{24} + \frac{3}{32}(1+4p)^2,$$

$$B_5 = -\frac{25}{24} + \frac{3}{50}(1+5p)^2, \quad B_6 = -\frac{25}{24} + \frac{3}{50}(2+5p)^2, \quad p \in \mathbb{Z}.$$

$$\sum_{j=1}^{7} \frac{1}{\Lambda_1(\mathbf{d}_j)\Lambda_2(\mathbf{d}_j)} = 1,$$

$$\sum_{j=1}^{7} \frac{\Lambda_1(\mathbf{d}_j) + \Lambda_2(\mathbf{d}_j)}{\Lambda_1(\mathbf{d}_j)\Lambda_2(\mathbf{d}_j)} = \sum_{j=1}^{7} \frac{1}{\Lambda_1(\mathbf{d}_j)} + \frac{1}{\Lambda_2(\mathbf{d}_j)} = -4,$$

$$\sum_{j=1}^{7} \frac{(\Lambda_1(\mathbf{d}_j) + \Lambda_2(\mathbf{d}_j))^2}{\Lambda_1(\mathbf{d}_j)\Lambda_2(\mathbf{d}_j)} = 16.$$

10. $\left\{\{-1,1\},\left\{-1,-\frac{3}{8}\right\},2\times\left\{-\frac{2}{3},\frac{7}{3}\right\},\left\{1,\frac{13}{8}\right\}2\times\{14,39\}\right\}$

$$V_{10}=\frac{4\sqrt{2}q_1^3}{3}+\frac{5q_1q_2^2}{2\sqrt{2}}+q_2^2q_3+\frac{1}{3}q_3^3,$$

$$\begin{aligned}I_1 &= 12p_2^4-27q_2^6-18q_2^4(q_1^2-4\sqrt{2}q_1q_3+2q_3^2)+4(6p_1^2-3p_3^2\\
&+16\sqrt{2}q_1^3-2q_3^3)(3p_3^2+2q_3^3)+12q_2^2(3p_3^2(\sqrt{2}q_1-4q_3)\\
&+12p_1p_3(q_1+\sqrt{2}q_3)-2q_3^2(12q_1^2+\sqrt{2}q_1q_3+2q_3^2))\\
&-12p_2q_2(2p_3(16q_1^2+3q_2^2+8\sqrt{2}q_1q_3-4q_3^2)+3\sqrt{2}p_1(q_2^2+4q_3^2))\\
&-12p_2^2(2p_3(2\sqrt{2}p_1+p_3)-4(q_2-q_3)q_3(q_2+q_3)-\sqrt{2}q_1(5q_2^2+8q_3^2)),\end{aligned}$$

$$\begin{aligned}I_2 &= 81q_2^8(2\sqrt{2}q_1+q_3)+216p_2p_3q_2^5(\sqrt{2}q_1+2q_3)+54q_2^6(p_2^2-3p_3^2\\
&+4\sqrt{2}q_1^3-24q_1^2q_3-6\sqrt{2}q_1q_3^2)+384p_2p_3q_1^2q_2(3p_2^2+8\sqrt{2}q_1^3+8q_1^2q_3\\
&-2\sqrt{2}q_1q_3^2)-72p_1^4(3p_3^2+2q_3^3)+144p_2p_3q_2^3(p_2^2+8q_1^2(2\sqrt{2}q_1+3q_3))\\
&+144p_1^3(\sqrt{2}p_2^2p_3+3\sqrt{2}p_2q_2q_3^2-3p_3q_2^2(q_1+\sqrt{2}q_3))-32(p_2^6\\
&+12p_2^4q_1^2q_3+12p_2^2q_1^3(\sqrt{2}p_3^2+4q_1q_3^2)+32q_1^6(3p_3^2+2q_3^3))\end{aligned}$$

$$
\begin{aligned}
&-12p_1^2(4p_2^4-6p_2p_3q_2(16q_1^2+9q_2^2+8\sqrt{2}q_1q_3-4q_3^2)+9q_2^4(2q_1^2\\
&+4\sqrt{2}q_1q_3+q_3^2)+32\sqrt{2}q_1^3(3p_3^2+2q_3^3)+12p_2^2(p_3^2+4q_2^2q_3\\
&-\sqrt{2}q_1(q_2^2-2q_3^2))+6q_2^2(9\sqrt{2}p_3^2q_1+2q_3^2(-6q_1^2+2\sqrt{2}q_1q_3+q_3^2)))\\
&-144q_2^4(p_2^2(7q_1^2+5\sqrt{2}q_1q_3+2q_3^2)+3q_1^2(3p_3^2-2q_3(-2q_1^2\\
&+2\sqrt{2}q_1q_3+q_3^2)))-48q_2^2(p_2^4(5\sqrt{2}q_1+4q_3)+4p_2^2q_1^2(8q_1^2\\
&+2\sqrt{2}q_1q_3+3q_3^2)+8q_1^3(9p_3^2q_1+q_3^2(-6\sqrt{2}q_1^2+4q_1q_3+\sqrt{2}q_3^2)))\\
&+6p_1(16\sqrt{2}p_2^4p_3+16p_2^2p_3(8q_1^3-6q_1q_2^2+3\sqrt{2}q_2^2q_3)\\
&+4p_2^3q_2(-16\sqrt{2}q_1^2+32q_1q_3+\sqrt{2}(3q_2^2+4q_3^2))+3p_3q_2^2(9\sqrt{2}q_2^4\\
&-64q_1^3(\sqrt{2}q_1+2q_3)-12q_2^2(2\sqrt{2}q_1^2+8q_1q_3+\sqrt{2}q_3^2))\\
&+12p_2q_2(-3\sqrt{2}p_3^2q_2^2+9q_1q_2^4+32q_1^3q_3^2+4q_2^2(4q_1^3+6q_1q_3^2+\sqrt{2}q_3^3))).
\end{aligned}
$$

Outline

Integrability problem for non-homogeneous potentials

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q_1, \dots, q_n)$$

V – non-homogenous

integrability in the Liouville sense of

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = \frac{\partial H}{\partial p_i} = p_i, \qquad \frac{\mathrm{d}}{\mathrm{d}t} p_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i}, \quad i = 1, \dots, n$$

decomposition

$$V = V_k + \dots + V_K, \quad K - k > 0,$$

where V_s are homogeneous functions of degree s i.e.

$$V_s(\lambda q_1, \dots, \lambda q_n) = \lambda^s V_s(q_1, \dots, q_n).$$

Integrability obstructions

- F. Mondejar, S. Ferrer, A. Vigueras, On the non-integrability of Hamiltonian systems with sum of homogeneous potentials, preprint, Universidad Politecnica de Cartagena
- H. Yoshida, Nonintegrability of the truncated Toda lattice Hamiltonian at any order, *Comm. Math. Phys.*, 116(4):529–538, 1988.

Integrability obstructions

Theorem (Mondejar, Yoshida)

Assume that Hamiltonian system with nonhomogeneous potential of the satisfies the following conditions:

1. *it admits a straight-line particular solution*

$$(\mathbf{q}(t), \mathbf{p}(t)) := (\varphi(t)\mathbf{d}, \dot{\varphi}(t)\mathbf{d}), \tag{2}$$

   where $\mathbf{d} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$*, and* $\varphi(t)$ *is a non-constant scalar function,*

2. *it is integrable in the Liouville sense with first integrals meromorphic in a connected neighbourhood* U *of the phase curve* $\mathbf{\Gamma}$ *corresponding to the above solution and functionally independent in* $U \setminus \mathbf{\Gamma}$*.*

Then the truncated systems given by the following Hamiltonian functions

$$H_k = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V_k(\mathbf{q}), \quad \text{and} \quad H_K = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V_K(\mathbf{q}), \qquad (3)$$

are integrable in the Liouville sense. Moreover, eigenvalues $(\lambda_1^{(\kappa)}, \ldots, \lambda_n^{(\kappa)})$ of Hessian matrices $V_\kappa''(\mathbf{d})$ with $\kappa = k, K$ satisfy the following condition: each pair $(\kappa, \lambda_j^{(\kappa)})$ for $\kappa = k, K$ and $j = 1, \ldots, n$ belongs to one of the items from the following list

case	κ	λ
1.	± 2	λ
2.	κ	$p + \frac{\kappa}{2}p(p-1)$
3.	κ	$\frac{1}{2}\left(\frac{\kappa-1}{\kappa} + p(p+1)\kappa\right)$
4.	3	$-\frac{1}{24} + \frac{1}{6}(1+3p)^2, \quad -\frac{1}{24} + \frac{3}{32}(1+4p)^2$ $-\frac{1}{24} + \frac{3}{50}(1+5p)^2, \quad -\frac{1}{24} + \frac{6}{25}(1+5p)^2$
5.	4	$-\frac{1}{8} + \frac{2}{9}(1+3p)^2$
6.	5	$-\frac{9}{40} + \frac{5}{18}(1+3p)^2, \quad -\frac{9}{40} + \frac{1}{10}(2+5p)^2$

case	κ	λ
7.	-3	$\frac{25}{24}-\frac{1}{6}(1+3p)^2,\quad \frac{25}{24}-\frac{3}{32}(1+4p)^2$
		$\frac{25}{24}-\frac{3}{50}(1+5p)^2,\quad \frac{25}{24}-\frac{3}{50}(2+5p)^2$
8.	-4	$\frac{9}{8}-\frac{2}{9}(1+3p)^2$
9.	-5	$\frac{49}{40}-\frac{5}{18}(1+3p)^2,\quad \frac{49}{40}-\frac{1}{10}(2+5p)^2$

Sketch of the proof

- an appropriate scaling of coordinates
- $$H_\kappa = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V_\kappa(\mathbf{q}), \quad \kappa \in \{k, K\}$$
- Morales-Ramis theorem for Hamilton system with homogeneous potential of degree κ

More conditions?

Remark

If V_k and V_K, $K-k>0$ are integrable, then usually

$$V = V_k + V_K$$

is not integrable!

Our theorem

Assume that:

1. the potential V is a sum of two homogeneous terms

$$V = V_k + V_K, \qquad k \in \mathbb{Z}^\star, \quad m = K - k \in \mathbb{N};$$

2. there exists a non-zero vector $\mathbf{d} \in \mathbb{C}^n$

$$V_k'(\mathbf{d}) = \alpha_k \mathbf{d}, \qquad V_K'(\mathbf{d}) = \alpha_K \mathbf{d},$$

   for a certain non-zero α_k and α_K.

3. $V_k''(\mathbf{d})$ and $V_K''(\mathbf{d})$ are simultaneously diagonalizable

$$\mathbf{C}^{-1} V_\kappa''(\mathbf{d}) \mathbf{C} = \alpha_\kappa \operatorname{diag}(\lambda_1^{(\kappa)}, \ldots, \lambda_n^{(\kappa)}),$$

   for a certain matrix $\mathbf{C}$.

4. the system is integrable in the Liouville sense.

Then, either $(\lambda_i^{(k)}, \lambda_i^{(K)}) = (\lambda_i^{(\kappa_1)}, \lambda_i^{(\kappa_2)})$, or $(\lambda_i^{(K)}, \lambda_i^{(k)}) = (\lambda_i^{(\kappa_1)}, \lambda_i^{(\kappa_2)})$ for $i = 1, \ldots, n$, where $\lambda_i^{(\kappa_1)}$ and $\lambda_i^{(\kappa_2)}$ belongs to an item of

case	$\lambda_i^{(\kappa_1)}$	$\lambda_i^{(\kappa_2)}$
1.	$\frac{1}{8\kappa_1}[4m^2r_1^2 - (\kappa_1-2)^2]$	$\frac{1}{8\kappa_2}[4m^2r_2^2 - (\kappa_2-2)^2]$
2.	$\frac{1}{8\kappa_1}[m^2(2l+1)^2 - (\kappa_1-2)^2]$	arbitrary
3.	$\frac{4m^2(3l+1)^2 - 9(\kappa_1-2)^2}{72\kappa_1}$	$\frac{4m^2(3p+1)^2 - 9(\kappa_2-2)^2}{72\kappa_2}$
4.	$\frac{4m^2(3l+1)^2 - 9(\kappa_1-2)^2}{72\kappa_1}$	$\frac{m^2(4p+1)^2 - 4(\kappa_2-2)^2}{32\kappa_2}$
5.	$\frac{4m^2(3l+1)^2 - 9(\kappa_1-2)^2}{72\kappa_1}$	$\frac{4m^2(5p+1)^2 - 25(\kappa_2-2)^2}{200\kappa_2}$
6.	$\frac{4m^2(5l+2)^2 - 25(\kappa_1-2)^2}{200\kappa_1}$	$\frac{4m^2(5p+1)^2 - 25(\kappa_2-2)^2}{200\kappa_2}$
7.	$\frac{4m^2(5l+2)^2 - 25(\kappa_1-2)^2}{200\kappa_1}$	$\frac{4m^2(3p+1)^2 - 9(\kappa_2-2)^2}{72\kappa_2}$

where $m = K - k$, $r_1, r_2 \in \mathbb{Q}$ satisfy

$$r_1 + r_2 = p + \frac{1}{2} \quad \text{or} \quad r_1 - r_2 = p + \frac{1}{2},$$

and l, p are integers

Sketch of the proof

- problem with particular solution.
- we look for a staight-line solution

$$\mathbf{q}(t) := \varphi(t)\mathbf{d}.$$

- sufficient condition for the existence:

$$V_k'(\mathbf{d}) = \alpha_k \mathbf{d} \quad \text{and} \quad V_K'(\mathbf{d}) = \alpha_K \mathbf{d},$$

with $\alpha_k, \alpha_K \in \mathbb{C}^\star$, i.e., V_k and V_K have a common Darboux point.

More conditions?

- particular solution $\mathbf{q} = \varphi\mathbf{d}$, $\mathbf{p} = \dot{\varphi}\mathbf{d}$

$$\ddot{\varphi} = -\alpha\varphi^{k-1} - \varphi^{K-1},$$
$$\dot{\varphi}^2 = 2e - \frac{2\alpha_k}{k}\varphi^k - \frac{2\alpha_K}{K}\varphi^K$$

- variational equations

$$\begin{bmatrix}\dot{Q}\\ \dot{P}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbb{I}\\ -\varphi^{k-2}V_k''(\mathbf{d}) - \varphi^{K-2}V_K''(\mathbf{d}) & \mathbf{0}\end{bmatrix}\begin{bmatrix}Q\\ P\end{bmatrix}$$

- we assume that $V_k''(\mathbf{d})$ and $V_K''(\mathbf{d})$ are simultaneously diagonalizable

$$\mathbf{C}^{-1}V_i''(\mathbf{d})\mathbf{C} = \alpha_i(\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \quad \text{for} \quad i = k, K,$$

- $$\ddot{\eta}_i = -\left(\alpha_k\lambda_i^{(k)}\varphi^{k-2} + \alpha_K\lambda_i^{(K)}\varphi^{K-2}\right)\eta_i, \quad i = 1, \ldots, n.$$

More conditions?

- rationalization by transformation $z = \varphi(t)$

$$\eta'' + p(z)\eta' + q(z)\eta = 0,$$
$$p = \frac{kK(\alpha_K z^m + \alpha_k)}{2z\left(k\alpha_K z^m + K\alpha_k\right)},$$
$$q = -\frac{kK(\alpha_K \lambda_j^{(K)} z^m + \alpha_k \lambda_j^{(k)})}{2z^2\left(k\alpha_K z^m + K\alpha_k\right)}.$$

- on energy level $e = 0$ after the next transformation $x = -\frac{k\alpha_K}{K\alpha_k} z^m$ we obtain

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}x^2} + \left(\frac{k+2m-2}{2mx} + \frac{1}{2(x-1)}\right)\frac{\mathrm{d}\eta}{\mathrm{d}x} + \left(-\frac{k\lambda_j^{(k)}}{2m^2x^2} + \frac{k\lambda_j^{(k)} - K\lambda_j^{(K)}}{2m^2x(x-1)}\right)\eta = 0. \quad (4)$$

Necessary integrability conditions for non-homogeneous potentials

hypergeometric equation with differences of exponents

$$\rho = \frac{1}{2m}\sqrt{(k-2)^2 + 8k\lambda_j^{(k)}}, \quad \sigma = \frac{1}{2}, \quad \tau = \frac{1}{2m}\sqrt{(K-2)^2 + 8K\lambda_j^{(K)}}.$$

Homogeneous deformations of radial potential

- class of potentials that are sum of two homogeneous parts with common Darboux points

$$V = \frac{a}{2}\left(\sum_{i=1}^{n} q_i^2\right)^s + V_\kappa(\mathbf{q}),$$

where $V_\kappa = V_\kappa(\mathbf{q})$ is a polynomial homogeneous potential of degree $\kappa, s \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$.

- in particular we will consider two cases: $s = 1$ (deformation of harmonic oscillator potential) and $s = -1/2$ (deformations of the Kepler problem)

Integrability conditions on perturbations of radial potential

Theorem

If Hamiltonian system with non-homogeneous potential V

$$V = \frac{a}{2}\left(\sum_{i=1}^{n} q_i^2\right)^s + V_\kappa(\mathbf{q}),$$

is meromorphically integrable in the Liouville sense and the highest order part of potential has degree homogeneity $\kappa \in \mathbb{Z}$*, then for a Darboux point values of* $2s \neq \kappa \in \mathbb{Z}$ *as well as nontrivial eigenvalues* λ_i *for* $i = 1, \ldots, n-1$ *belong to the following list*

case	$2s$	λ_i
1.	$\dfrac{2(\kappa r-1)}{2r+1}$	$\dfrac{(2p+1-2r)^2(\kappa-2s)^2-(\kappa-2)^2}{8\kappa}$
2.	$\kappa \pm \frac{\kappa+2}{2l}$	arbitrary
3.	arbitrary	$\dfrac{(2l+1)^2(\kappa-2s)^2-(\kappa-2)^2}{8\kappa}$
4.	$\kappa \pm \dfrac{3(\kappa+2)}{6l-1}$	$\dfrac{4(3p+1)^2(\kappa-2s)^2-9(\kappa-2)^2}{72\kappa}$
		$\dfrac{(4p+1)^2(\kappa-2s)^2-4(\kappa-2)^2}{32\kappa}$
		$\dfrac{4(5p+1)^2(\kappa-2s)^2-25(\kappa-2)^2}{200\kappa}$
		$\dfrac{4(5p+2)^2(\kappa-2s)^2-25(\kappa-2)^2}{200\kappa}$

case	$2s$	λ_i
5.	$\kappa \pm \dfrac{2(\kappa+2)}{4l-1}$	$\dfrac{4(3p+1)^2(\kappa-2s)^2-9(\kappa-2)^2}{72\kappa}$
6.	$\kappa \pm \dfrac{5(\kappa+2)}{10l-3}$	$\dfrac{4(3p+1)^2(\kappa-2s)^2-9(\kappa-2)^2}{72\kappa}$
		$\dfrac{4(5p+2)^2(\kappa-2s)^2-25(\kappa-2)^2}{200\kappa}$
7.	$\kappa \pm \dfrac{5(\kappa+2)}{10l-1}$	$\dfrac{4(5p+1)^2(\kappa-2s)^2-25(\kappa-2)^2}{200\kappa}$
		$\dfrac{4(3p+1)^2(\kappa-2s)^2-9(\kappa-2)^2}{72\kappa}$

where l, p are integers and r rational. There is no obstructions on $2s$ and λ in the case when $\kappa = -2$.

Deformations of oscillator potential, $n = 2$, $\kappa = 3$

- $V = \frac{\alpha}{2}(q_1^2 + q_2^2) + V_3$
- integrable potentials V_3

Case	Potential	$\{\lambda_1, \lambda_2, \lambda_3\}$
1	$(q_1 + \mathrm{i} q_2)^l (q_1 - \mathrm{i} q_2)^{3-l}, \; l = 0, 1$	w. D. p.
2	q_1^3	$\{0\}$
3	$q_1^3 + c q_2^3/3$	$\{0, 0, 2\}$
4	$q_1^2 q_2/2 + q_2^3$	$\{1/3, 5, 5\}$
5	$q_1^2 q_2/2 + 8 q_2^3/3$	$\{1/8, 15, 15\}$
6	$\mathrm{i}\sqrt{3} q_1^3/18 + q_1^2 q_2/2 + q_2^3$	$\{1/3, 10/3, 15\}$

- only items 2 and 3 of new table are admissible and both gives the set

$$\left\{ \frac{1}{6} p(p+1) \mid p \in \mathbb{Z} \right\}$$

- only integrable deformations of potentials 1, 2, 3, 4 and 6 are admissible

Deformations of oscillator potential, $n = 2$, $\kappa = 3$

$$V = (q_1 - \mathrm{i}q_2)^3 + \frac{\alpha}{2}(q_1^2 + q_2^2), \qquad I_1 = (p_1 - \mathrm{i}p_2)^2 + \alpha(q_1 - \mathrm{i}q_2)^2,$$

$$I_2 = 3p_1^4 - 12\mathrm{i}p_1^3p_2 + 3p_2^4 - 4\alpha^2p_2^2q_1 - 4\mathrm{i}p_1p_2(-3p_2^2 + \alpha(3(q_1 - \mathrm{i}q_2)^2 + \alpha(q_1 + \mathrm{i}q_2))) - 6\alpha p_2^2(q_1 - \mathrm{i}q_2)^2 - \alpha^4q_2^2 - \alpha^3(p_2^2 - 2\mathrm{i}(q_1 - \mathrm{i}q_2)^2q_2) - 2p_1^2(9p_2^2 + \alpha(-3(q_1 - \mathrm{i}q_2)^2 - 2\mathrm{i}\alpha q_2)),$$

$$V = (q_1 - \mathrm{i}q_2)^2(q_1 + \mathrm{i}q_2) + \frac{\alpha}{2}(q_1^2 + q_2^2),$$

$$I = p_1^2 - 6\mathrm{i}p_1p_2 - 5p_2^2 + (q_1 - \mathrm{i}q_2)(\alpha q_1 - \mathrm{i}(5\alpha + 8q_1)q_2 - 8q_2^2),$$

$$V = q_1^3 + \frac{\alpha}{2}(q_1^2 + q_2^2), \qquad I = p_2^2 + \alpha q_2^2,$$

$$V = q_1^3 + \frac{c}{3}q_2^3 + \frac{\alpha}{2}(q_1^2 + q_2^2), \qquad I = 3(p_2^2 + \alpha q_2^2) + 2cq_2^3,$$

$$V = \frac{1}{2}q_1^2q_2 + q_2^3 + \frac{\alpha}{2}(q_1^2 + q_2^2),$$

$$I = 8p_1(p_2q_1 - p_1q_2) + 48\alpha^2q_1^2 + q_1^4 + 4q_1^2q_2^2 + 8\alpha(3p_1^2 + 2q_1^2q_2)$$

- deformation of potential 6 is non-integrable – II order variational eqs.

Deformations of oscillator potential, $n = 3$, $\kappa = 3$

$$V = (q_1^2 + q_2^2)q_3 + \frac{1}{3}q_3^3 + \frac{\alpha}{2}(q_1^2 + q_2^2 + q_3^2), \quad I_1 = p_2 q_1 - p_1 q_2,$$

$$I_2 = 6p_2 p_3 q_2(q_1^2 + q_2^2 + 3q_3(\alpha + q_3)) + (q_1^2 + q_2^2)(q_1^2 + q_2^2 + 3q_3(\alpha + q_3))^2$$
$$+ 3p_2^2(3p_3^2 + q_1^2(\alpha + 4q_3)) + 3p_1^2(3p_3^2 + q_2^2(\alpha + 4q_3))$$
$$+ 6p_1 q_1(-p2q2(\alpha + 4q_3) + p_3(q_1^2 + q_2^2 + 3q_3(\alpha + q_3))),$$

$$V = (q_1^2 + q_2^2 + 2q_3^2)q_3 + \frac{\alpha}{2}(q_1^2 + q_2^2 + q_3^2), \quad I_1 = p_2 q_1 - p_1 q_2,$$

$$I_2 = 4p_1(p_3 q_1 - p_1 q_3) + 4p_2(p_3 q_2 - p_2 q_3) + (q_1^2 + q_2^2)(q_1^2 + q_2^2 + 4q_3^2)$$
$$+ \alpha(p_1^2 + p_2^2 - 2p_3^2 - 8q_3^3) + \alpha^2(q_1^2 + q_2^2 - 2q_3^2).$$