A class of Hermitian manifolds with integrable geodesic flows

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0. Introduction

Liouville manifold is, roughly speaking, a manifold equipped with a "Liouville metric" (or that of "Liouville-Stäckel type"). A simple type of such metric has the form

$$g = \sum_{i=1}^{n} (-1)^{n-i} \prod_{k \neq i} (f_k(x_k) - f_i(x_i))\, dx_i^2$$

for some coordinates $(x_1, \ldots, x_n)$ and functions $f_i(x_i)$ in one variable.

- The geodesic flow possesses n independent first integrals which are fiberwise quadratic forms.
- The geodesic flow is completely integrable.

Typical examples are:

- Manifolds of constant curvature
- Quadratic hypersurfaces in the Euclidean spaces

Kähler-Liouville manifold is a Hermitian version of the notion of Liouville manifold. It is a Kähler manifold ($\dim_{\mathbb{C}} = n$) whose geodesic flow possesses n first integrals which are *fiberwise Hermitian forms*. General features are:

- The geodesic flow is (generally) completely integral.
- It contains real n-dim Liouville manifold as a totally geodesic submanifold.

Typical example is $\mathbb{CP}^n$ with the Fubini-Study metric. Many toric varieties admit such structures.

On the Kähler condition:

- In the theory of K-L manifolds, the Kähler condition works in an effective way to determine local and global structure.
- But the Kähler condition is *a priori* unrelated to the integrability of the geodesic flow.
- Moreover, if E is the Hamiltonian of the geodesic flow and F_i are first integrals, then $E' = E + \sum_i \epsilon_i F_i$ (ϵ_i: small) corresponds to a Hermitian metric and the corresponding geodesic flow is still integrable, but the metric is generally non-Kähler.

Therefore it is natural to study **Hermite-Liouville manifold**, the same definition as K-L manifold, but the metric is not necessarily Kähler, only Hermitian. Another natural example of H-L manifold arises from h**-projective equivalence** of Kähler metrics.

Two riemannian metrics g and $\tilde{g}$ on a manifold are said to be *projectively equivalent* if their geodesic orbits coincide; namely,

$$\tilde{\nabla}_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X.$$

Levi-Civita locally determined the forms of such metrics. As a result, in the most general case, those metrics become Liouville metrics.

Later, **Matveev and Topalov** developed the global theory of projectively equivalent metrics; invariant description of the first integrals of the geodesic flow, construction of a hierarchy of the pairs of such metrics $(g, \tilde{g})$, $(g_A, \tilde{g}_A)$, $(g_{A^2}, \tilde{g}_{A^2})$, $\ldots$.

Topalov also considered a Kähler analogue of it; h-projective equivalence.

Two Kähler metrics on a complex manifold are said to be *h-projectively equivalent* if the classes of curves $\gamma(t)$ satisfying

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = a(t)\dot{\gamma}(t) + b(t)\, J\, \dot{\gamma}(t)$$

coincides; namely,

$$\tilde{\nabla}_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X - \phi(JX)JY - \phi(JY)JX.$$

Topalov found first integrals and showed that, under some nondegeneracy condition, those manifolds become Kähler-Liouville manifolds.

He also considered a hierarchy $(g, \tilde{g})$, $(g_A, \tilde{g}_A)$, $\ldots$, as in the real case, but in this case

- g_A, $\tilde{g}_A$ are Hermitian, but not Kähler in general.
- $(g_A, \tilde{g}_A)$ are not h-projectively equivalent, but only satisfy its variant;

$$\tilde{\nabla}_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X + \phi(Q^{-1}X)QY + \phi(Q^{-1}Y)QX,$$

where Q is a skew symmetric, nondegenerate $(1,1)$-type tensor. In this case, the manifold is a *Hermite-Liouville manifold* and the geodesic flow is still completely integrable.

The third example of H-L manifold is the following one.

Let E_0 be the Hermite-type ellipsoid in $\mathbb{C}^{n+1}$ given by

$$E_0: \quad \sum_{i=0}^{n} \frac{|z_i|^2}{a_i} = 1 \qquad (a_0 > \cdots > a_n > 0)$$

and let

$$\rho : E_0 \left(\subset \mathbb{C}^{n+1} - \{0\}\right) \to \mathbb{CP}^n$$

be the natural $U(1)$-bundle. Then $\mathbb{CP}^n$ with the natural Hermitian metric induced from ρ is a H-L manifold (non-Kähler) and the geodesic flow is completely integrable.

The aim of this talk is:

- To construct H-L manifolds over $\mathbb{CP}^n$ from simple materials (functions on circles and diffeomorphisms of circles)
- To show which ones are mutually isomorphic, which ones are Kähler, and which ones correspond to the examples quoted above, among the constructed H-L manifolds.

Plan of this talk.

1. Liouville manifolds (Review)
2. Kähler-Liouville manifolds (Review)
3. h-projective equivalence of Kähler metrics
4. Hermite-Liouville manifolds over $\mathbb{CP}^n$
5. Examples and conjecture

1. Liouville manifolds

By definition, Liouville manifold is a pair of a Riemannian manifold (M, g), $\dim M = n$, and an n-dim. vector space $\mathcal{F}$ of functions on the cotangent space T^*M which satisfy the following conditions.

(1) For every $F \in \mathcal{F}$ and $p \in M$, $F_p := F|_{T_p^*M}$ is a quadratic form.

(2) F_p, $F \in \mathcal{F}$, are simultaneously normalizable for each $p \in M$.

(3) $\mathcal{F}$ contains the Hamiltonian E of the geodesic flow.

(4) The Poisson bracket $\{F, H\}$ vanishes for every $F, H \in \mathcal{F}$.

(5) $\mathcal{F}_p := \{\ F_p\,; F \in \mathcal{F}\ \}$ is n-dimensional at some point $p \in M$.

A Liouville manifold $(M, g; \mathcal{F})$ is called *proper* if $F_p = 0$ for some $F \in \mathcal{F} - \{0\}$ and $p \in M$, then $dF_\xi \neq 0$ for some $\xi \in T_p^*M$.

The notion of *rank* is defined for proper Liouville manifold:

$$1 \leq \mathrm{rank}(M, g; \mathcal{F}) \leq \dim M$$

- Maximal rank (rank $= \dim M$) & M: compact $\Rightarrow$ a finite cover of M is diffeomorphic to a torus.
- Rank one Liouville manifolds are completely classified. They are diffeomorphic to S^n (type A), $\mathbb{RP}^n$ (type B), or $\mathbb{R}^n$ (type C, D).
- It is conjectured that a finite cover of compact L. m. of rank r is diffeomorphic to a product of r spheres (of various dimensions).

Liouville manifolds of rank one, type (B), are classified by means of a circle $\mathbb{R}/l\mathbb{Z}$ $(l > 0)$ with the standard metric dt^2 and projective classes of $n-1$ functions $[f_1(t)], \ldots, [f_{n-1}(t)]$ on it, called *core of type (B)*. They satisfy the following conditions (for some representatives f_i).

1. There are constants $0 < \beta_1 < \cdots < \beta_{n-1} < l/2$ such that $f_m(\pm\beta_m) = 0$, $f_m(t) > 0$ for $-\beta_m < t < \beta_m$, and $f_m(t) < 0$ for $\beta_m < t < l - \beta_m$.
2. $f'_m(\beta_m) < 0$.
3. $f_m(t) = f_m(-t)$ for any $t \in \mathbb{R}/l\mathbb{Z}$.
4. $f_1(t) < \cdots < f_{n-1}(t)$ for any $t \in \mathbb{R}/l\mathbb{Z}$.

Theorem. *There is a one-one correspondence between the isomorphism classes of proper Liouville manifolds of rank one, type (B) and the isomorphism classes of cores of type (B).*

Definition of isomorphisms:

$$(M, g; \mathcal{F}) \simeq (M', g'; \mathcal{F}')$$
$$\Longleftrightarrow \exists \phi : (M, g) \underset{\to}{\simeq} (M', g') \quad \text{with} \quad \phi_* \mathcal{F} = \mathcal{F}'.$$

$$(\mathbb{R}/l\mathbb{Z}; [f_1], \ldots, [f_{n-1}]) \simeq (\mathbb{R}/\tilde{l}\mathbb{Z}; [\tilde{f}_1], \ldots, [\tilde{f}_{n-1}])$$
$$\Longleftrightarrow \quad \tilde{l} = l, \quad [\tilde{f}_i(t)] = [f_i(t)] \quad (1 \le i \le n-1),$$
$$\text{or}$$
$$\tilde{l} = l, \quad [\tilde{f}_i(t)] = [-f_{n-i}(l/2 - t)] \quad (1 \le i \le n-1).$$

To simplify the notation, for

$$\mathcal{C} = (\mathbb{R}/l\mathbb{Z}; [f_1], \ldots, [f_{n-1}])$$

we write

$$\mathcal{C}^r = (\mathbb{R}/l\mathbb{Z}; [f_1^r], \ldots, [f_{n-1}^r]),$$

wehre

$$f_i^r(t) = -f_{n-i}(l/2 - t) \quad (1 \leq i \leq n-1).$$

Then,

$$\mathcal{C}_1 \simeq \mathcal{C}_2 \quad \Longleftrightarrow \quad \mathcal{C}_2 = \mathcal{C}_1 \quad \text{or} \quad \mathcal{C}_2 = \mathcal{C}_1^r$$

From a core of type (B) one can construct a Liouville manifold of type (B) as follows. Put $\beta_0 = 0$, $\beta_n = l/2$, and define positive numbers $\alpha_1, \dots, \alpha_n$ by

$$\int_{\beta_{i-1}}^{\beta_i} \frac{dt}{\sqrt{(-1)^{i-1} f_1(t) \dots f_{n-1}(t)}} = \frac{\alpha_i}{4} \, .$$

Define the C^∞ mapping

$$\mathbb{R}/\alpha_i\mathbb{Z} \longrightarrow \begin{cases} [\beta_{i-1}, \beta_i] & (2 \le i \le n-1) \\ [-\beta_1, \beta_1] & (i = 1) \\ [\beta_{n-1}, l - \beta_{n-1}] & (i = n) \end{cases}$$

$(w_i \mapsto t)$ by

$$\left(\frac{dt}{dw_i}\right)^2 = (-1)^{i-1} f_1(t) \dots f_{n-1}(t),$$
$$t(0) = \beta_i, \quad t(\alpha_i/4) = \beta_{i-1}.$$

Put

$$R = \prod_{i=1}^{n} (\mathbb{R}/\alpha_i \mathbb{Z}) = \{(w_1, \dots, w_n)\},$$

and define the involutions σ_i, $1 \leq i \leq n-1$, and τ on R by

$$\sigma_i(x) = (w_1, \dots, w_{i-1}, -w_i, \frac{\alpha_{i+1}}{2} - w_{i+1}, w_{i+2}, \dots, w_n),$$
$$\tau(x) = (w_1 + \frac{\alpha_1}{2}, -w_2, \dots, -w_n).$$

They generate a group G isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, and the quotient space $N = R/G$ with a natural differentiable structure is diffeomorphic to $\mathbb{RP}^n$.

Define the functions $f_{ik} \in C^\infty(\mathbb{R}/\alpha_i\mathbb{Z})$ by

$$f_{ik}(w_i) = f_k(t(w_i)), \qquad 1 \leq k \leq n-1,\ 1 \leq i \leq n,$$

and the matrix-valued function $[b_{ij}(w_i)]_{1\leq i,j\leq n}$ by

$$b_{ij} = b_{ij}(w_i) = \begin{cases} (-1)^i \prod_{k\neq j} f_{ik}(w_i) & (1 \leq j \leq n-1), \\ (-1)^{i+1} \prod_k f_{ik}(w_i) & (j = n). \end{cases}$$

Then by the formula

$$\sum_{j=1}^{n} b_{ij}(w_i) F_j = (\partial/\partial w_i)^2, \qquad 1 \leq i \leq n,$$

one obtains well-defined symmetric 2-tensor fields $F_1, \ldots, F_n$ on N. Also, F_n turns out to be positive definite at any point. Thus, putting

$$\mathcal{F} = \mathrm{Span}\{F_1, \ldots, F_n\},$$

one gets a Liouville manifold $(N, g; \mathcal{F})$ over N whose energy function (the Hamiltonian of the geodesic flow) is equal to $F_n/2$.

Examples: (1) $\mathbb{RP}^n$ of constant curvature 1 corresponds to

$$l = \pi, \quad f_i(t) = (\cos t)^2 - c_i \quad (1 \leq i \leq n-1)$$

where $1 > c_1 > \cdots > c_{n-1} > 0$ are arbitrary.

(2) E being the ellipsoid $\displaystyle\sum_{i=0}^{n} \frac{x_i^2}{a_i} = 1 \quad (a_0 > \cdots > a_n > 0)$, the Riemannian manifold $E/\{\pm\text{identity}\}$ corresponds to

$$l = \frac{1}{2} \times \text{ the length of the ellipse } \frac{x_0^2}{a_0} + \frac{x_n^2}{a_n} = 1,$$

$$f_i(t) = (\cos s(t))^2 - \frac{a_i - a_n}{a_0 - a_n} \qquad (1 \leq i \leq n-1),$$

$$\frac{ds}{dt} = \frac{1}{\sqrt{a_0(\cos s)^2 + a_n(\sin s)^2}}$$

2. Kähler-Liouville manifolds

A Kähler-Liouville manifold is, by definition, a pair of a Kähler manifold (M, g), $\dim_{\mathbb{C}} M = n$, and an n-dimensional real vector space $\mathcal{F}$ of functions on the cotangent bundle T^*M which satisfies the following conditions.

(1) For every $F \in \mathcal{F}$ and $p \in M$, $F_p := F|_{T_p^*M}$ is a Hermitian form.

(2) F_p, $F \in \mathcal{F}$, are simultaneously normalizable for each $p \in M$.

(3) $\mathcal{F}$ contains the Hamiltonian E of the geodesic flow.

(4) The Poisson bracket $\{F, H\}$ vanishes for every $F, H \in \mathcal{F}$.

(5) $\mathcal{F}_p := \{ F_p \, ; F \in \mathcal{F} \}$ is n-dimensional at some point $p \in M$.

Under some nondegeneracy conditions (proper, type (A)), we have the following results.

- There is an n-dim. Lie algebra of infinitesimal automorphisms $\mathcal{Y}$ of (M, g) such that it is abelian and $\{Y, F\} = 0$ for any $Y \in \mathcal{Y}$ and $F \in \mathcal{F}$. In particular, the geodesic flow ot (M, g) is completely integrable with $\mathcal{Y}$ and $\mathcal{F}$.
- If M is compact, then $\mathcal{Y}$ generates a n-torus action on M, and with this action M becomes a toric variety.

As in the real case, we have the notion of *rank*, and:

- If M is compact and the rank of $(M, g; \mathcal{F})$ is one, then M is isomorphic to $\mathbb{CP}^n$ as a toric variety.
- If M is compact, then taking a "real part" of $(M, g; \mathcal{F})$, one has a Liouville manifold of the same rank.

In compact, rank one case, the real part is a Liouville manifold of rank one, type (B); the corresponding core of type (B) has the following form.

$$(\mathbb{R}/l\mathbb{Z}; [v(t)-c_1], \dots, [v(t)-c_{n-1}])\,,$$

where $1 > c_1 > \cdots > c_{n-1} > 0$ and $v(t) \in C^\infty(\mathbb{R}/l\mathbb{Z})$ satisfies

(1) $v(-t) = v(t)$.

(2) $v(0) = 1$, $v(l/2) = 0$.

(3) $v'(t) < 0$ if $0 < t < l/2$.

(4) $-v''(0) = v''(l/2) = c_*$.

(5) $v'(\beta_i) = -\sqrt{2c_*c_i(1-c_i)}$, where $\beta_i = v^{-1}(c_i) \in (0, l/2)$, $1 \le i \le n-1$.

We call such a core "of special kind".

Then we have

Theorem. *There is a one-one correspondence between the isomorphism classes of Kähler-Liouville manifolds of rank one and the isomorphism classes of type (B) cores of special kind.*

In this case, if

$$\mathcal{C} = (\mathbb{R}/l\mathbb{Z}; [v(t) - c_1], \ldots, [v(t) - c_{n-1}]\},$$

then

$$\mathcal{C}^r = (\mathbb{R}/l\mathbb{Z}; [v^r(t) - c_1^r], \ldots, [v^r(t) - c_{n-1}^r]\},$$

where

$$v^r(t) = 1 - v(l/2 - t), \qquad c_i^r = 1 - c_{n-i}.$$

How to construct K-L manifold of rank one from a core of special kind:

1. Construct the corresponding Liouville manifold $(N, g; \mathcal{F})$.
2. Define vector fields $X_0, \dots, X_n$ on N by the formula

$$X_i = \frac{\text{grad}\,\left(\prod_k (v_k - c_i)\right)}{c_* \prod_{\substack{0 \le m \le n \\ m \ne i}} (c_m - c_i)}, \qquad 0 \le i \le n, \quad c_0 = 1,\ c_n = 0,$$

where $v_k(w_k) = v(t(w_k))$ as explained before. They satisfy

$$[X_i, X_j] = 0 \quad (\forall i, j) \quad \text{and} \quad \sum_{i=0}^{n} X_i = 0.$$

3. Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n = \{[u_0, \dots, u_n]\}$ be the natural projection. Then one can find a diffeomorphism $\phi : N \to \mathbb{RP}^n$ such

that

$$\phi_*(X_i) = \pi_*\left(u_i(\partial/\partial u_i)\right), \qquad 0 \le i \le n.$$

4. Let $\mathbb{CP}^n$ be the complex projective space with the homogeneous coordinates $[u_0, \dots, u_n]$ whose real part is $\mathbb{RP}^n$. The torus $U(1)^n = U(1)^{n+1}/U(1)$ naturally acts on $\mathbb{CP}^n$:

$$((\lambda_0, \dots, \lambda_n), [u_0, \dots, u_n]) \mapsto [\lambda_0 u_0, \dots, \lambda_n u_n], \quad |\lambda_i| = 1.$$

5. Then the vector fields X_i extends to $\mathbb{CP}^n$ so that they are invariant under the torus action. Clearly, $Y_i = JX_i$, $0 \le i \le n$, generate the torus action.
6. Also, each $F \in \mathcal{F}$ is extended to the whole $\mathbb{CP}^n$ so that they are "hermitian" and invariant by the torus action.

7. The similarly extended metric g is a Kähler metric, and one obtains a Kähler-Liouville manifold $(\mathbb{CP}^n, g; \mathcal{F})$.

Example. If $l = \pi$, $v(t) = (\cos t)^2$ $(1 > c_1 > \cdots > c_{n-1} > 0$ are arbitrary), then one obtains $\mathbb{CP}^n$ with the Fubini-Study metric.

3. h-projective equivalence of Kähler metrics

Let g and $\tilde{g}$ be projectively equivalent riemannian metrics on a manifold M;

$$\tilde{\nabla}_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X.$$

Let A be the (1,1)-type tensor defined by $\tilde{g}(\cdot,\cdot) = \det(A)^{-1} g(A^{-1}\cdot,\cdot)$. Then

- $K_c(\dot{\gamma}(t)) = \det(A - cI)\, g((A - cI)^{-1}\,\dot{\gamma}(t), \dot{\gamma}(t))$ is constant along any geodesic $\gamma(t)$ of (M, g) for any $c \in \mathbb{R}$.

Thus, under certain nondegeneracy conditions, (M, g) becomes a Liouville manifold, and so is $(M, \tilde{g})$.

Also,

- Putting $g_A(\cdot,\cdot) = g(A\,\cdot,\cdot)$, the two metrics $(g_A, \tilde{g}_A)$ are again projectively equivalent.

Therefore one obtains a hierarchy of pairs of projectively equivalent metrics; $(g, \tilde{g})$, $(g_A, \tilde{g}_A)$, $(g_{A^2}, \tilde{g}_{A^2})$, $\ldots$, or more generaly, $(g_{u(A)}, \tilde{g}_{u(A)})$ for suitable analytic function $u(t)$ in one variable.

Let g and $\tilde{g}$ be h-projectively equivalent Kähler metrics on a complex manifold M;

$$\tilde{\nabla}_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X - \phi(JX)JY - \phi(JY)JX.$$

Let A be the $(1,1)$-type tensor defined by $\tilde{g}(\cdot,\cdot) = \det(A)^{-1/2} g(A^{-1}\cdot,\cdot)$.

Then we have

- $AJ = JA$.
- $K_c(\dot{\gamma}(t)) = \det(A - cI)^{1/2}\, g\left((A - cI)^{-1}\,\dot{\gamma}(t), \dot{\gamma}(t)\right)$ is constant along any geodesic $\gamma(t)$ of (M, g) for any $c \in \mathbb{R}$.

Put $\mathcal{F} = \mathrm{Span}\,\{K_c^* \mid c \in \mathbb{R}\}$, $K_c^* = K_c$, regarded as functions on T^*M by identifying TM and T^*M with g.

Let

$$h_1 \geq \cdots \geq h_n$$

be the eigenfunctions of A (as $\mathbb{C}$-linear endomorphisms).

We assume the following nondegeneracy condition:

$$h_1(p) > \cdots > h_n(p) \quad \text{and} \quad dh_i \neq 0 \ (\forall i) \quad \text{at some point } p \in M.$$

Then we have (assuming M: compact)

Theorem (K.-Topalov). *$(M, g; \mathcal{F})$ is a Kähler-Liouville manifold of rank one. In particular, M is biholomorphic to $\mathbb{CP}^n$ and the the geodesic flow of (M, g) is completely integrable. Conversely, for any K-L manifold of rank one there is another K-L manifold $(M, \tilde{g}; \tilde{\mathcal{F}})$ such that g and $\tilde{g}$ are h-projectively equivalent.*

Related works

• The notion of *Hamiltonian 2-forms* given by Apsotolov, Calderbank, and Gauduchon treats essentially the same thing as h-projective equivalence (motivations are different, though).

• The notion of Conformal (Yano) Killing tensor, which are recently studied in the context of generalized Kerr spacetime, also seems to treat the same thing as h-projective equivalence.

Correspondence of cores:

If
$$\mathcal{C} = (\mathbb{R}/l\mathbb{Z};\ [h(t) - c_1], \dots, [h(t) - c_{n-1}])$$

is the core of $(M, g; \mathcal{F})$, then the core of h-projectively equivalent $(M, \tilde{g}; \tilde{\mathcal{F}})$ is given by

$$\tilde{\mathcal{C}} = (\mathbb{R}/\tilde{l}\mathbb{Z};\ [\tilde{h}(\tilde{t}) - \tilde{c}_1], \dots, [\tilde{h}(\tilde{t}) - \tilde{c}_{n-1}]),$$

where, $a > 0$, $\gamma > 0$ being any constants,

$$\tilde{h}(\tilde{t}(t)) = \frac{a\, h(t)}{(a-1)\, h(t) + 1}, \quad \tilde{c}_i = \frac{a\, c_i}{(a-1)\, c_i + 1}$$
$$\frac{d\tilde{t}}{dt} = \frac{\sqrt{a\gamma}}{(a-1)\, h(t) + 1}, \quad \tilde{l} = \int_0^l \frac{\sqrt{a\gamma}}{(a-1)\, h(t) + 1}\, dt,$$

or, the same formulas with $h(t)$ and c_i replaced by $h^r(t)$ and c_i^r respectively.

In this case, we say that *two cores $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are h-projectively equivalent and that $\phi : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}/\tilde{l}\mathbb{Z}$ ($t \mapsto \tilde{t}$) gives the h-projective equivalence of the cores.* The map ϕ induces a holomorphic diffeomorphism

$$\Phi : M \to \tilde{M}$$

and the two metric g and $\Phi^* \tilde{g}$ are h-projectively equivalent on M.

If the first formulas are satisfied, then $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are called h-projectively equivalent *in positive order.*

4. Hermite-Liouville manifolds over $\mathbb{CP}^n$

The definition of Hermite-Liouville manifolds is the same as that of K-L manifolds, except that the metric is not necessarily Kähler, but only Hermitian.

We shall construct a Hermite-Liouville manifold by mean of two cores of type (B), one is a general kind and the other is a special kind;

$$\mathcal{C} = (\mathbb{R}/l\mathbb{Z}; [f_1(t)], \ldots, [f_{n-1}(t)]),$$
$$\tilde{\mathcal{C}} = (\mathbb{R}/\tilde{l}\mathbb{Z}; [h(s) - c_1], \ldots, [h(s) - c_{n-1}]),$$

and a diffeomorphism

$$\phi : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}/\tilde{l}\mathbb{Z} \quad (t \mapsto s)$$

such that $ds/dt > 0$ and

$$\phi(0) = 0, \quad \phi(-t) = -\phi(t), \quad \phi(\beta_i) = \tilde{\beta}_i,$$

where $0 < \beta_i < l/2$ and $0 < \tilde{\beta}_i < \tilde{l}/2$ are defined as $f_i(\beta_i) = 0$ and $h(\tilde{\beta}_i) = c_i$ respectively.

The H-L manifold $(M, g; \tilde{\mathcal{F}})$ to be constructed possesses the following properties.

- M is biholomorphic to $\mathbb{CP}^n$.
- (M, g) admits an n-dimensional abelian Lie algebra $\mathcal{Y}$ of infinitesimal automorphisms so that $\{Y, F\} = 0$ for any $Y \in \mathcal{Y}$ and $F \in \tilde{\mathcal{F}}$. In particular, the geodesic flow of (M, g) is completely integrable with respect to the first integrals $\mathcal{Y}$ and $\tilde{\mathcal{F}}$.

Remark. The results in the case where ϕ =Identity were obtained by Igarashi-K. in [1].

The construction goes as follows:

1. First, construct Liouville manifold $(N, g; \mathcal{F})$ from the core of general kind.
$$N = R/\sim, \quad R = \prod_{i=1}^{n} \mathbb{R}/\alpha_i\mathbb{Z}, \quad \text{etc..}$$

2. Next, construct L. m. $(\tilde{N}, \tilde{g}, \mathcal{H})$ from the core of special kind and identify it as $\mathbb{RP}^n \subset \mathbb{CP}^n$ so that it can be complexified.
$$N \underset{\rightarrow}{\simeq} \mathbb{RP}^n \subset \mathbb{CP}^n.$$

3. The diffeomorphism $\phi : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}/\tilde{l}\mathbb{Z}$ yields the diffeomorphisms

$$\begin{array}{ccc} \mathbb{R}/\alpha_i\mathbb{Z} & \longrightarrow & \mathbb{R}/\tilde{\alpha}_i\mathbb{Z} \\ \Big\downarrow & & \Big\downarrow \\ [\beta_{i-1}, \beta_i] & \xrightarrow{\ \phi\ } & [\tilde{\beta}_{i-1}, \tilde{\beta}_i] \end{array}$$

and hence

$$R \underset{\rightarrow}{\simeq} \tilde{R}, \qquad \Phi : N \underset{\rightarrow}{\simeq} \tilde{N}.$$

4. Instead of complexifying $(\tilde{N}, \tilde{g}, \mathcal{H})$, we complexify $\Phi_*(N, g; \mathcal{F})$ by using the scheme

$$N \underset{\rightarrow}{\simeq} \tilde{N} \underset{\rightarrow}{\simeq} \mathbb{RP}^n \subset \mathbb{CP}^n$$

Then we have the following theorems.

Theorem. *The constructed H-L manifold is Kähler if and only if the core $\mathcal{C}$ is also of special kind and the two cores $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are h-projectively equivalent with the diffeomorphism $\phi : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}/\tilde{l}\mathbb{Z}$.*

In this case the resulting H-L manifold is isomprphic to the K-L manifold constructed with $\mathcal{C}$.

Let us consider two triplets $\quad (k = 1, 2)$:

$$C_k = (\mathbb{R}/l_{(k)}\mathbb{Z}; [f_{(k)1}(t)], \ldots, [f_{(k)n-1}(t)]),$$
$$\tilde{C}_k = (\mathbb{R}/\tilde{l}_{(k)}\mathbb{Z}; [h_{(k)}(s) - c_1], \ldots, [h_{(k)}(s) - c_{n-1}]),$$
$$\phi_k : \mathbb{R}/l_{(k)}\mathbb{Z} \to \mathbb{R}/\tilde{l}_{(k)}\mathbb{Z} \quad (t \mapsto s)$$

and the resulting H-L manifolds $(M_k, g_k; \mathcal{F}_k)$ $(k = 1, 2)$.

Then we have

Theorem. *The Hermite-Liouville manifolds* $(M_k, g_k; \mathcal{F}_k)$ $(k = 1, 2)$ *are mutually isomorphic if and only if*

(1) C_1 *and* C_2 *are isomorphic.*

(2) $\tilde{C}_1$ *and* $\tilde{C}_2$ *are h-projectively equivalent with a diffeomorphism* $\phi : \mathbb{R}/\tilde{l}_{(1)}\mathbb{Z} \to \mathbb{R}/\tilde{l}_{(2)}\mathbb{Z}$.

(3) $\phi_2 = \phi \circ \phi_1$ *(in order) or* $\phi_2 \circ r = \phi \circ \phi_1$ *(reversed order).*

Here, $r : \mathbb{R}/l_{(1)}\mathbb{Z} \to \mathbb{R}/l_{(2)}\mathbb{Z}$ $(l_{(1)} = l_{(2)} = l)$ is given by $r(t) = l/2 - t$.

5. Examples and conjecture

Example 1. If g, $\tilde{g}$ are h-projectively equivalent Kähler metrics on M, then the geodesic flows of (M, g_A) and $(M, \tilde{g}_A)$ are also completely integrable. However:

- The Hermitian metrics g_A and $\tilde{g}_A$ are no longer Kähler in general.
- The Levi-Civita connections of g_A and $\tilde{g}_A$ satisfy only a variant of h-projective equivalence, i.e.,

  $$\tilde{\nabla}_X Y - \nabla_X Y = \phi(X)Y + \phi(Y)X + \phi(Q^{-1}X)QY + \phi(Q^{-1}Y)QX,$$

  where Q is a skew symmetric, nondegenerate $(1,1)$-type tensor on M.

Theorem (K.-Topalov). Let g and $\tilde{g}$ be two Hermitian metrics on a complex manifold M of $\dim_{\mathbb{C}} M = n$. Suppose that they satisfy the variant of h-projective equivalence for some $(1,1)$-tensor Q, and suppose that the nondegeneracy conditions stated before hold on a neighborhood U of some point $p \in M$. Then:

(1) $(M, g; \mathcal{F})$ is a Hermite-Liouville manifold on U ($\mathcal{F}$ being similarly defined as before).

(2) There is an n-dim. abelian Lie algebra $\mathcal{Y}$ of infinitesimal automorphisms of (M, g) on U such that $\mathcal{Y}$ and $\mathcal{F}$ are elementwise commutative with respect to the Poisson bracket. In particular, the geodesic flow of (M, g) is completely integrable on U.

Theorem. *The constructed H-L manifold is identical with one that appeared in the variant of h-projective equivalence if and only if the corresponding triplet is of the form*

$$(\psi^*\mathcal{C}, \mathcal{C}, \psi)$$

where $\mathcal{C} = (\mathbb{R}/l\mathbb{Z}; [h(t)-c_1], \ldots, [h(t)-c_{n-1}])$, *is any core of special kind,* $l' > 0$ *is any, and* $\psi : \mathbb{R}/l'\mathbb{Z} \to \mathbb{R}/l\mathbb{Z}$ *is any diffeomorphism such that* $\psi(0) = 0, \psi(-t) = -\psi(t)$, *and*

$$\psi^*\mathcal{C} = (\mathbb{R}/l'\mathbb{Z}; [\psi^*h(t) - c_1], \ldots, [\psi^*h(t) - c_{n-1}]).$$

More precisely, for any h-projective equivalence $\phi : \mathcal{C} \to \tilde{\mathcal{C}}$, there are $\tilde{l}' > 0$ and diffoemorphisms $\tilde{\psi} : \mathbb{R}/\tilde{l}'\mathbb{Z} \to \mathbb{R}/\tilde{l}'\mathbb{Z}$ and $\phi' : \mathbb{R}/l'\mathbb{Z} \to \mathbb{R}/\tilde{l}'\mathbb{Z}$ so that the diagram

$$\begin{array}{ccc} \mathbb{R}/l\mathbb{Z}\ (\mathcal{C}) & \xrightarrow{\phi} & \mathbb{R}/\tilde{l}\mathbb{Z}\ (\tilde{\mathcal{C}}) \\ \psi \uparrow & & \uparrow \tilde{\psi} \\ \mathbb{R}/l'\mathbb{Z}\ (\psi^*\mathcal{C}) & \xrightarrow{\phi'} & \mathbb{R}/\tilde{l}'\mathbb{Z}\ (\tilde{\psi}^*\tilde{\mathcal{C}}) \end{array}$$

commutes, and (ϕ', ϕ) induces the variant of h-projective equivalence.

Example 2. $(M, g; \mathcal{H})$ a K-L manifold of rank one with the core

$$\tilde{\mathcal{C}} = (\mathbb{R}/l\mathbb{Z}; [h(t) - c_1], \ldots, [h(t) - c_{n-1}])$$

$H_1, \ldots, H_n = 2E$ be a suitable basis of $\mathcal{H}$. Then,

$$2E' = 2E + \sum_{i=1}^{n-1} \epsilon_i H_i \qquad (|\epsilon_i| \text{ sufficiently small})$$

is still positive definite on each fiber, and the corresponding metric g' defines a H-L manifold $(M, g'; \mathcal{H})$. The corresponding triplet is of the form $(\mathcal{C}, \tilde{\mathcal{C}}, \text{Identity})$, where $\mathcal{C} = (\mathbb{R}/l\mathbb{Z}; [f_1(t)], \ldots, [f_{n-1}(t)])$ is given by

$$f_i(t) = \frac{h(t) - c_i}{1 + \epsilon_i(h(t) - c_i)}\,, \quad (1 \leq \imath \leq n - 1)\,.$$

Example 3. Let E_0 be the Hermite-type ellipsoid in $\mathbb{C}^{n+1}$ given by

$$E_0 : \quad \sum_{i=0}^{n} \frac{|z_i|^2}{a_i} = 1 \qquad (a_0 > \cdots > a_n > 0)$$

and let

$$\rho : E_0 \left(\subset \mathbb{C}^{n+1} - \{0\}\right) \to \mathbb{C}P^n$$

be the natural $U(1)$-bundle. Then, the bundle ρ naturally induces a Hermitian metric on $\mathbb{C}P^n$, and with this metric $\mathbb{C}P^n$ becomes a H-L manifold. The corresponding triplet $(\mathcal{C}, \tilde{\mathcal{C}}, \phi)$ is:

- $\tilde{\mathcal{C}} = (\mathbb{R}/\pi\mathbb{Z}; [\cos^2 t - c_1], \ldots, [\cos^2 t - c_{n-1}])$, i.e., the core of the $\mathbb{C}P^n$ with the Fubini-Study metric. $c_i = \dfrac{a_i - a_n}{a_0 - a_n}$ $(1 \leq i \leq n-1)$.

- $\phi : \mathbb{R}/l\mathbb{Z} \to \mathbb{R}/\pi\mathbb{Z} \quad (t \mapsto s)$ is given by

$$\phi(0) = 0, \qquad \frac{ds}{dt} = \frac{1}{\sqrt{a_0 \cos^2 s + a_n \sin^2 s}},$$

  and l is a half of the length of the ellipse $x_0^2/a_0 + x_n^2/a_n = 1$.
- $\mathcal{C} = \phi^* \tilde{\mathcal{C}} = (\mathbb{R}/l\mathbb{Z}; [\cos^2 s(t) - c_1], \ldots, \cos^2 s(t) - c_{n-1}]$. Namely, it is the core of the (real) ellipsoid

$$\sum_{i=0}^{n} \frac{x_i^2}{a_i} = 1 \qquad \text{in} \quad \mathbb{R}^{n+1}$$

- Thus the triplet is of the form $(\phi^* \tilde{\mathcal{C}}, \tilde{\mathcal{C}}, \phi)$; the same form appeared in the variant of h-projective equivalence.

Conjecture:

"Let $(M, g; \mathcal{F})$ be a Hermite-Liouville manifold over $\mathbb{CP}^n$ which satisfies certain nondegeneracy condition (proper and type (A)). Suppose that (M, g) admits an n-dim. abelian Lie algebra of infinitesimal automorphisms which elementwise commutes with $\mathcal{F}$. Then $(M, g; \mathcal{F})$ is isomorphic to one of the constructed H-Ls."

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