$\mathbf{1+3}$ decomposition of quadratic Killing-Stäckel tensors in curved spaces

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Objectives

Consider a four dimensional metric (euclidean or minkowskian), in coordinates: t, x^i, with a single Killing vector ∂_t. Let us look for quadratic Killing-Stäckel (K-S) tensors $\mathcal{S}$ invariant under ∂_t. Our aim is to devise some *local* constructive approach which reduces the four dimensional Killing equations to a system of PDE immersed in a genuinely three dimensional metric. It is what we call the $1+3$ decomposition of quadratic K-S tensors.

Geometrical setting

Let us consider the metric

$$g = \frac{\epsilon}{V}\,(dt+\theta)^2 + V\,\gamma_{ij}\,dx^i\,dx^j, \qquad \epsilon = \pm 1$$

$$V = V(x) \qquad\qquad \theta = \theta_i(x)\,dx^i$$

having at least the Killing vector ∂_t.

The hamiltonian flow (phase space coordinates : $t, x^i,\ P_0, P_i$) is generated by

$$H = \frac{1}{2}\left(\epsilon\,V\,P_0^2 + \frac{1}{V}\,||\Pi||^2_\gamma\right) \qquad \Pi_s = P_s - \theta_s\,P_0$$

and gives for evolution equations

$$\dot{P}_0 = 0 \quad \dot{P}_i = \left(\frac{H}{V} - \epsilon\,P_0^2\right)\partial_i\,V + \frac{P_0}{V}\Pi^s\partial_i\theta_s - \frac{1}{2V}(\partial_i\gamma^{st})\,\Pi_s\,\Pi_t$$

Let us define the coordinates change in phase space

$$(t,\,x^i,\,P_0,\,P_i) \to (t,\,x^i,\,\Pi_0,\,\Pi_i) : \qquad \Pi_0 = P_0 \qquad \Pi_i = P_i - \theta_i\,P_0$$

which is *non-canonical* since

$$\Omega = d\Pi_0 \wedge dt + d\Pi_i \wedge dx^i + d(\theta\,\Pi_0)$$

The flow for $\Pi^i = \gamma^{is}\,\Pi_s$ becomes

$$\dot\Pi^i = \left(\frac{H}{V} - \epsilon\,\Pi_0^2\right)\partial^i V + \frac{1}{V}\,\Pi_0\,F_{is}\,\Pi^s - \frac{1}{V}\,\Sigma^i_{st}\,\Pi^s\,\Pi^t$$

where $F = d\theta$ and the Σ are the Christoffel symbols of γ.

Using $\mathcal{L}_{\partial_t}\mathcal{S} = 0$ and writing the K-S tensor as

$$\mathcal{S} = A_{ij}(x)\,\Pi^i\,\Pi^j + 2B_i(x)\,\Pi_0\,\Pi^i + C(x)\,\Pi_0^2 + 2D(x)\,H$$

its conservation gives the following set of PDE:

$$B^s\,\partial_s V = 0$$

$$\nabla_{(k}\,A_{ij)} = 0 \qquad \nabla_{(i}\,B_{j)} + F_{s(i}\,\gamma^{st}\,A_{j)t} = 0$$

$$\nabla_i\,C = 2(\epsilon\,A_{is}\,V\,\nabla^s V + F_{is}\,B^s) \qquad \nabla_i\,D = -A_{is}\,\nabla^s V$$

where ∇ is the Levi-Civita connection for the metric γ.

`Trivial remarks`:

1. The functions C and D are defined up to an additive constant.

2. Avoid $A = \gamma$ and $B = 0$ since it leads to $\mathcal{S} = 0$.

Integrability

To achieve integrability let us assume that there is some extra Killing vector of the metric, purely spatial, i.e. of the form $K^i\,\partial_i$ such that $\mathcal{L}_K\,\mathcal{S} = 0$. Then if $\mathcal{S}$ is *irreducible* we achieve Liouville integrability with 4 conserved quantities in involution for the Poisson bracket:

$$P_0 \qquad\qquad K^i\,P_i \qquad\qquad H \qquad\qquad \mathcal{S}$$

One can set $B^i = F\,K^i$ and the previous PDE system reduces to

$$\nabla_{(k}\,A_{ij)} = 0 \qquad K_{(i}\,\nabla_{j)}\,F + F_{s(i}\,\gamma^{st}\,A_{j)t} = 0$$

$$\nabla_i\,C = 2(\epsilon\,A_{is}\,V\,\nabla^s\,V + F_{is}\,B^s) \qquad \nabla_i\,D = -A_{is}\,\nabla^s\,V$$

Walker and Penrose theorem

Proved in:

`M. Walker and R. Penrose`
Commun. Math. Phys. **18** (1970) 265

Proposition: *Any Petrov type D vacuum metric admits a quadratic conformal K-S tensor*

$$\nabla_{(a}Q_{bc)} = Q_{(a}g_{bc)} \qquad Q_{bc} = Q_{cb}$$

$$Q_{ab} = (A^2 + B^2)[k_a\,l_b + k_b\,l_a + \tfrac{1}{2}\,g_{ab}] \qquad (\Psi_2)^{-1/3} = \text{const}\,(A + iB)$$

where Ψ_2 is the single non-vanishing component of the Weyl tensor and $(m,\,\overline{m},\,k,\,l)$ is a complex null tetrad.

Furthermore, if $Q_a = \nabla_a q$ then $K_{ab} = Q_{ab} - q\,g_{ab}$ is an irreducible K-S tensor if the metric has three or less Killing vectors.

`W. Kinnersley`
J. Math. Phys. **10** (1969) 1195

has classified all type D vacuum metrics. They all happen to have at least 2 commuting Killing vectors. With the notable exception of the C-metrics, their geodesic flows are integrable as a result of this theorem.

`Question`: how much is left over by this theorem?

The Multi-Centre metrics

These are euclidean and Ricci-flat metrics:

$$\gamma = \sum_{i=1}^{3} (dX^i)^2 \qquad\qquad F \equiv d\theta = * \, dV \quad \text{(monopole equation)}$$

The monopole equation implies that $\Delta V = 0$ in the flat 3-dim. space γ.

Let us summarize some basic facts:

1. The metric is hyperkäler with the triplet of covariantly constant complex structures

$$\Omega_i = (dt + \theta) \wedge dX^i - \frac{V}{2} \epsilon_{ijk} \, dX^j \wedge dX^k$$

This implies Ricci-flatness.

2. The Killing vector ∂_t is tri-holomorphic (tri-H).

It was shown in

C. P. Boyer and J. D. Finley
J. Math. Phys. **23** (1982) 1126

that for an hyperkähler geometry any Killing vector K is either tri-H (translational)

$$\mathrm{tri-H}: \qquad \mathcal{L}_K \, \Omega_i = 0$$

or holomorphic H (rotational):

$$\mathrm{H}: \quad \mathcal{L}_K \, \Omega_1 = -\Omega_2 \qquad \mathcal{L}_K \, \Omega_2 = \Omega_1 \qquad \mathcal{L}_K \, \Omega_3 = 0$$

Euclidean Weyl tensor

One first defines a tetrad and the basis of self-dual 2-forms

$$g=(e_0)^2+(e_1)^2+(e_2)^2+(e_3)^2 \qquad \lambda_i^{\pm}=e_0\wedge e_i \pm \frac{1}{2}\epsilon_{ijk}\, e_j\wedge e_k$$

The self-dual components of the curvature are decomposed as follows

$$R_i^+ = A_{ij}\,\lambda_j^+ + B_{ij}\,\lambda_j^- \qquad R_i^- = (B^t)_{ij}\,\lambda_j^+ + C_{ij}\,\lambda_j^-$$

which gives for the self-dual components of the Weyl tensor two 3×3 *real* matrices

$$W^+ = C - \frac{1}{3}\,(\mathrm{tr}\, C)\,\mathbb{I} \qquad W^- = A - \frac{1}{3}\,(\mathrm{tr}\, A)\,\mathbb{I}$$

For the Multi-Centre one has

$$F = *\, dV \quad \text{(monopole equation)} \quad \Longrightarrow \quad W^+ = 0$$

Since W^- is traceless either the Petrov type is general (type I) or it is special (type D) with 2 equal eigenvalues.

`Simplifying aspects`:

1. The full metric is under control of a single harmonic function V.

2. The metric γ being flat, all of its quadratic Killing tensors are given by tensor products of its Killing vectors.

`G. H. Katzin and J. Levine`
Tensor **16** (1965) 97

The construction of K-S tensors for the Multi-Centre metrics had received a lot of attention, particularly for Taub-NUT metric for which $V = a + \dfrac{1}{||\vec{r}||}$, for which a Runge-Lenz-like vector was found in

G. W. Gibbons and N. S. Manton
Nucl. Phys. B **274** (1986) 183

and

L. G. Feher and P. A. Horváthy
Phys. Lett. B **183** (1987) 182

and gave rise to the determination of the bound states and diffusion states of Schrödinger equation in the Taub-NUT background.

Other results, to be mentioned later, were derived in

G. W. Gibbons and P. J. Ruback
Commun. Math. Phys. **115** (1988) 267

However these analyses did not lead to any systematic resolution of the general problem for the Multi-Centre metrics.

Let us proceed in two steps, according to the nature of the extra spatial Killing vector:

$$\begin{cases} u(1)_H : & K^i\,\partial_i = x\,\partial_y - y\,\partial_x = \partial_\phi \\ u(1)_{triH} : & K^i\,\partial_i = \partial_z \end{cases}$$

as described in

G. Valent
Commun. Math. Phys. **244** (2004) 571

(I): One extra H spatial Killing vector

The metric is under control of $V(\rho, z)$ using cylindrical coordinates and the monopole equation gives $G(\rho, z)$.

1) Solving for A

The most general Killing tensor $A(\Pi, \Pi) = A_{ij}\,\Pi^i\,\Pi^j$ is known since it is a bilinear form in the Killing vectors coded in

$$\Pi_i \qquad\qquad \vec{L} = \vec{r} \wedge \vec{\Pi}.$$

Imposing the constraint $\mathcal{L}_K\,A_{ij} = 0$ reduces it to

$$A(\Pi, \Pi) = \alpha(L_x^2 + L_y^2) + \beta\,L_z^2 + c^2\,\Pi_z^2 + \gamma(\vec{\Pi} \wedge \vec{L})_z + \delta\,\Pi_z\,L_z + a\,\vec{\Pi}^2$$

Some simplifications occur:

1. Since the piece L_z^2 is reducible we may choose at will its coefficient: we take $\beta = \alpha$.

2. The piece in front of a, after integration of the equations for (C, D) gives $a(\vec{\Pi}^2 - 2HV + V^2\,P_0^2)$ which vanishes due to energy conservation.

3. The relation

$$K_{(i}\,\nabla_{j)}F + A_{s(i}\epsilon_{j)st}\,\partial_t V = 0 \qquad\qquad \ddagger$$

implies both

$$K^i\,\partial_i F = 0 \qquad\qquad A_{is}\,K_s = a(x)\,K_i.$$

The last constraint gives $\delta = 0$.

So we end up with:

$$A(\Pi, \Pi) = \alpha\,\vec{L}^2 + c^2\,\Pi_z^2 + \gamma(\vec{\Pi} \wedge \vec{L})_z$$

2) Solving for F

The equations for F are first

$$K_{(i}\,\nabla_{j)}F + A_{s(i}\epsilon_{j)st}\,\partial_t V = 0$$

and they become coupled PDE in two variables

$$\partial_\rho F = -(\alpha\, z + \gamma/2)\,\partial_\rho V + \alpha\rho\,\partial_z V$$

$$\rho\,\partial_z F = -(\alpha z^2 - c^2 + \gamma\, z)\partial_\rho V + \rho(\alpha z + \gamma/2)\,\partial_z V$$

First write down their integrability conditions, which result in a decoupled PDE for V and then solve for F.

3) Solutions for V

One finds 3 cases:

1. $\alpha = 1 \qquad c \neq 0$

$$V = v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-} \qquad\qquad r_\pm = \sqrt{\rho^2 + (z \pm c)^2}$$

2. $\alpha = 1 \qquad c = 0$

$$V = v_0 + \frac{m}{r} + v_1\, z \qquad\qquad r = \sqrt{\rho^2 + z^2}$$

3. $\alpha = c = 0 \qquad b = 1$

$$V = v_0 + \frac{m}{r} + v_1\,\frac{z}{r^3}$$

The remaining equations for (C, D) are integrable and one gets the full K-S tensor. We will not give the explicit form of the K-S tensors which can be found in my CMP article.

1) The two-Centre metric

$$V = v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-}$$

`Observations:`

1. Generically it is Petrov type I and type D iff $v_0 = 0$.

2. K-S tensor first obtained by Gibbons and Ruback by separating Hamilton-Jacobi equation in spheroidal coordinates.

3. For $v_0 = 0$ and $m_1 = m_2$ the isometries are enhanced to $su(2)_{tH} \oplus u(1)_H$ (Bianchi IX with 4 Killing vectors). One gets Eguchi-Hanson metric

T. Eguchi and A. J. Hanson
Physics Lett. B **74** (1978) 249

4. For $v_0 = 0$ and $m_1 = -m_2$ the isometries are enhanced to $su(1,1)_{tH} \oplus u(1)_H$ (Bianchi VIII with 4 Killing vectors). One gets Gegenberg-Das metric

J. D. Gegenberg and A. Das
Gen. Rel. Grav. **16** (1984) 817

5. In both cases the K-S becomes *reducible.*

2) Uniform field breaking of Taub-NUT

$$V = v_0 + \frac{m}{r} + v_1\, z$$

`Observations:`

1. Generically it is Petrov type I and type D iff (either $v_0 = 0$ or $v_1 = 0$).

2. For $m = 0$ one can take $v_0 = 0$, $v_1 = 1$. One recovers a bi-axial Bianchi II metric with 4 Killing vectors. Acting with these isometries produces a triplet of irreducible K-S tensors, generating a first W-algebra (described later on).

3. For $v_1 = 0$ we get Taub-NUT with isometries $su(2)_H \oplus u(1)_{tH}$ hence Bianchi IX. Acting with the $su(2)_H$ produces a triplet of irreducible K-S tensors generating a second W-algebra (described also later on).

`More on Taub-NUT:`

(Work by `Gibbons and Manton`, paper already quoted).

1. There are 4 Killing vectors, with linear conserved quantities

$$\vec{J} = \vec{L} + \Pi_0 \frac{\vec{r}}{r} \qquad \text{and} \qquad \Pi_0$$

2. The triplet of K-S tensors is best written

$$\vec{\mathcal{S}} = \vec{\Pi} \wedge \vec{J} - \frac{\vec{r}}{r}(H - a\,\Pi_0^2)$$

3. Despite the 4 Killing vectors the K-S tensors are *irreducible*.

3) Dipolar breaking of Taub-NUT

$$V = v_0 + \frac{m}{r} + v_1 \frac{z}{r^3}$$

`Observations:`

1. Generically it is Petrov type I and type D iff either $v_0 = 0$ or $v_1 = 0$.

2. For $m = 0\, v_0 = 0$ one recovers a bi-axial Bianchi III metric with 4 Killing vectors. The K-S tensor becomes *reducible*.

3. For $v_1 = 0$ we get again Taub-NUT but the K-S tensor $\vec{L}^2$ is reducible since it is equal to $\vec{J}^2 - \Pi_0^2$.

(II): One extra tri-H spatial Killing vector

We have, using cylindrical coordinates: $(V,G)=(V,G)(x,y)$.

One has to consider successively four cases for $A(\Pi,\Pi)$:

$$(1):\ L_z^2+c^2\,\Pi_x^2 \qquad (2):\ L_z^2 \qquad (3):\ \Pi_y\,L_z \qquad (4):\ \alpha\,\Pi_y^2+2\beta\,\Pi_y\,\Pi_x$$

Using the complex variable $w=x+iy$ let us define two complex functions

$$\mathcal{V}=V+iG \qquad\qquad \mathcal{D}=D+iF$$

From the monopole equation we have

$$\partial^2_{w\overline{w}}\,V=\partial^2_{w\overline{w}}\,G=0 \qquad\qquad \partial_{\overline{w}}\,\mathcal{V}=0$$

The function $\mathcal{D}$ is constrained by

$$\partial_w\,\mathcal{D}=-(A_{11}+A_{22})\,\partial_w\,V$$

$$\partial_{\overline{w}}\,\mathcal{D}=-(A_{11}-A_{22}+2i\,A_{12})\,\partial_w\,V$$

Here too the integrability for $\mathcal{D}$ implies a differential equation for the potential in the sole variable w.

One gets the following results:

`1) First case:`

• For $c \in \mathbb{R}$ one defines $w = c\, \cos(\theta - i\chi)$ and then

$$V = v_0 + 2\frac{(\mu \sinh\chi\ \cosh\chi + \nu\ \sin\theta\ \cos\theta)}{\cosh^2\chi - \cos^2\theta} \qquad (v_0, \mu, \nu) \in \mathbb{R}^3$$

`Observations:`

1. The metric is *always* Petrov type I.

2. For $v_0 = \nu = 0$ and $\mu = 1/2$ the metric has 3 Killing vectors with Lie algebra Bianchi VII_0.

3. The K-S tensor is always irreducible. It was first given in

`A. N. Aliev, M. Hortaçsu, J. Kalayci and Y. Nutku`
Class. Quantum Grav. **16** (1999) 631

• For $c \in i\mathbb{R}$ one changes $c \to i\, c$, and defines $w = c\, \sinh(\theta + i\chi)$. Then

$$V = v_0 + 2\frac{(\mu \sinh\chi\ \cosh\chi - \nu\ \sin\theta\ \cos\theta)}{\cosh^2\chi - \sin^2\theta} \qquad (v_0, \mu, \nu) \in \mathbb{R}^3$$

`Observations:`

1. The metric is *always* Petrov type I.

2. For $v_0 = \mu = 0$ and $\nu = -1/2$ the metric has 3 Killing vectors with Lie algebra Bianchi VI_0.

3. The K-S tensor is always irreducible.

`2) Second case:`

Using polar coordinates one gets

$$V = v_0 + \frac{\mu}{r^2}\sin(2\phi) \qquad\qquad (v_0, \mu) \in \mathbb{R}^2$$

`Observations:`

1. The metric is *always* Petrov type I.

2. For $v_0 = 0$ the metric has 3 Killing vectors with Lie algebra Bianchi VI_0. One recovers the same metric and K-S tensor as in the first case above.

`3) Third case:`

Using square parabolic coordinates one gets

$$V = v_0 + 2\frac{(\mu\,\xi + \nu\,\eta)}{\xi^2 + \eta^2} \qquad (v_0,\, \mu,\, \nu) \in \mathbb{R}^3$$

`Observations:`

1. The metric is Petrov type I and type D iff $v_0 = 0$.

2. For $v_0 = 0$ the metric has 3 Killing vectors with Lie algebra Bianchi II (get nothing new).

`3) Fourth case:`

This case corresponds to Bianchi II, *always* Petrov type D. No new K-S tensor appears.

W-algebras

Algebras closing non-linearly under Poisson bracket; see

`J. de Boer, F. Harmsze and T. Tjin`
Phys. Rep. **272** (1996) 139

1) Darboux metric

A super-integrable model:

$$g = u(du^2 + dv^2) \qquad H = \frac{P_u^2 + P_v^2}{2u} \qquad \{H, P_v\} = 0$$

Conserved linear observable (isometry):

$$P_v$$

Conserved quadratic observables (K-S tensors):

$$\mathcal{S}_1 = P_u\, P_v - v\, H \qquad \mathcal{S}_2 = P_v(vP_u - uP_v) - \frac{v^2}{2}\, H$$

Isometry action:

$$\{P_v, \mathcal{S}_1\} = -H \qquad \{P_v, \mathcal{S}_2\} = \mathcal{S}_1$$

Non-linear closing:

$$\{\mathcal{S}_1, \mathcal{S}_2\} = -2\, P_v^3$$

2) Bianchi II

G. Valent and A. ben Yahia
Class. Quantum Grav. **24** (2007) 255

$$H = \frac{1}{2}\left(s\,P_0^2 + \frac{1}{s}\,(P_y - x\,P_0)^2 + \frac{1}{s}(P_s^2 + P_x^2)\right)$$

Conserved linear observables:

$$L_1 = P_0 \quad L_2 = P_x - y\,P_0 \qquad L_3 = P_y \quad L_4 = x\,P_y - y\,P_x - \frac{1}{2}(x^2 - y^2)P_0$$

Lie algebra:

$$\{L_2, L_3\} = L_1 \qquad \{L_4, L_2\} = -\,L_3 \qquad \{L_4, L_3\} = L_2$$

$$\text{Casimir :} \qquad \mathcal{T} = L_2^2 + L_3^2 - 2\,L_1\,L_4$$

Conserved quadratic observables:

$$\mathcal{S}_2 = sP_0(P_y - xP_0) - P_s\,P_x + x\,H$$

$$\mathcal{S}_3 = P_0(x\,P_s - s\,P_x) - P_s\,P_y + y\,H$$

$$\mathcal{S}_4 = s(P_x^2 + P_y^2 - yP_x - xP_y) + P_s(xyP_0 - xP_x - yP_y) + \frac{1}{2}(x^2 + y^2)H$$

Isometries action:

$$\{L_2, \mathcal{S}_2\} = H \qquad \{L_4, \mathcal{S}_2\} = -\mathcal{S}_3 \qquad \{L_3, \mathcal{S}_3\} = H$$

$$\{L_4, \mathcal{S}_3\} = \mathcal{S}_2 \qquad \{L_2, \mathcal{S}_4\} = \mathcal{S}_2 \qquad \{L_3, \mathcal{S}_4\} = \mathcal{S}_3$$

Non-linear closing:

$$\{\mathcal{S}_2, \mathcal{S}_3\} = 2L_1\,\mathcal{T} \qquad \{\mathcal{S}_4, \mathcal{S}_2\} = 2L_2\,\mathcal{T} \qquad \{\mathcal{S}_4, \mathcal{S}_3\} = 2L_3\,\mathcal{T}$$

3) Taub-NUT (Bianchi IX)

(K-S tensor found by `Gibbons and Manton` paper quoted).

`G. W. Gibbons and P. J. Ruback`
CMP **115** (1988) 267

`L. Féher and P. Horvathy`
Phys. Lett. B **183** (1987) 182

$$H = \frac{1}{2}\left(V\,P_0^2 + \frac{1}{V}\,\vec{P}^2\right) \qquad\qquad V = a + \frac{1}{||\vec{r}||}$$

Conserved linear observables:

$$P_0 \qquad\qquad J_i = \epsilon_{ijk}\,x_j\,\Pi_k + \frac{x_i}{r}\,P_0 \qquad i = 1, 2, 3$$

Conserved quadratic observables:

$$\mathcal{S}_i = \epsilon_{ijk}\,\Pi_j\,J_k - \frac{x_i}{r}(H - a\,P_0^2)$$

$$\vec{L}^2 = \vec{J}^2 + P_0^2 \qquad \text{fully reducible}$$

Lie algebra:

$$\{J_i, J_j\} = -\epsilon_{ijk}\,J_k \qquad\qquad \{P_0, J_i\} = 0$$

Isometries action:

$$\{P_0, \mathcal{S}_i\} = 0 \qquad\qquad \{J_i, \mathcal{S}_j\} = -\epsilon_{ijk}\,\mathcal{S}_k$$

Non-linear closing:

$$\{\mathcal{S}_i, \mathcal{S}_j\} = -(a^2\,P_0^2 - 2a\,H)\,\epsilon_{ijk}\,J_k$$

fairly similar to the Runge-Lenz vector for the classical Kepler problem.

Summary

V:	$v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-}$	$v_0 + \frac{m}{r} + v_1\, z$	$v_0 + \frac{m}{r} + v_1 \frac{z}{r^3}$
P-type:	I	I	I
Special:	$\underbrace{B\,VIII;\ B\,IX}$	$\underbrace{B\,II;\ B\,IX}$	B III
Special P:	D	D	D

$A(\Pi, \Pi)$:	$L_z^2 + c^2\Pi_x^2$	L_z^2	$\Pi_y\, L_z$	$\alpha\Pi_y^2 + 2\beta\Pi_x\Pi_y$
P-type:	I	I	I	D (B II)
Special:	$\underbrace{B\,VI_0;\ B\,VII_0}$	B VI_0	$\underbrace{B\,VII_0;\ B\,II}$	$\times$
Special P:	I	I	I D	

Let us give the list of the possible Bianchi metrics:

class A :	II	VI_0	VII_0	$VIII$	IX
class B :	III	IV	V	VI_h	VII_h

It was proved in

`G. Valent arXiv: gr-qc/1102.4538v1`

that in the class B only the Bianchi type III allows for hyperkähler metrics.

Conclusion: Penrose and Walker theorem insufficient since most metrics are Petrov type I.

Outlook

The approach presented here could be applied to the following problems:

1. The Le Brun metrics:

C. R. Le Brun
J. Diff. Geom. **34** (1991) 223

They are (euclidean) scalar-flat Kähler metrics: a monopole equation on $\gamma = \mathbb{H}^3$ instead of $\mathbb{R}^3$. Some preliminary results were obtained in

G. W. Gibbons and C. M. Warnick
J. Geom. Phys. **57** (2007) 2286

2. The (Weyl self-dual) euclidean Einstein metrics

M. J. Calderbank and H. Pedersen
J. Diff. Geom. **60** (2002) 485-521

A linearization on the hyperbolic plane with two commuting Killing vectors.

3. The minkowskian Ernst equations. They describe the Stationary and Axisymmetric Vacuum solutions (hence two commuting Killing vectors) of Einstein equations. An early attempt

C. M. Cosgrove
J. Phys. A **11** (1978) 2417