Lagrangian Bundles and Affinely Flat Geometry

Daniele Sepe

University of Leicester

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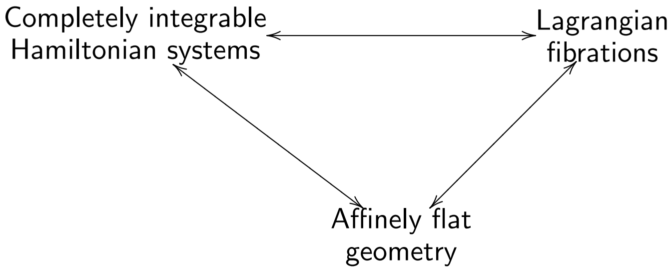
Main idea


Completely integrable Hamiltonian systems
Lagrangian fibrations
Affinely flat geometry

Main motivation

Definition

A completely integrable Hamiltonian system (CIHS) on (M, ω) is a map $\mathbf{f} = (f_1, \ldots, f_n) : (M, \omega) \to \mathbb{R}^n$ satisfying

- $\{f_i, f_j\} = 0$ for all i, j - the components are in involution;
- $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_n \neq 0$ almost everywhere on M - the components are functionally independent.

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(hopeless?) Aims

- Classify CIHS (topologically, smoothly, symplectically, up to orbital equivalence, etc.);
- Construct interesting examples.

Lagrangian fibrations

Definition

Let (M, ω) be a $2n$-dimensional symplectic manifold. A (proper) surjective smooth map $\pi : (M, \omega) \to B$ is a Lagrangian fibration if the generic fibre $F_b = \pi^{-1}(b)$ is a Lagrangian submanifold of (M, ω).

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Examples

Cotangent bundle $(\mathrm{T}^*B, \Omega) \to B$, focus-focus singularity, Kodaira-Thurston fibering over $\mathbb{T}^2$.

Action-angle coordinates

Liouville-Arnold theorem $\Rightarrow$ existence of action-angle coordinates trivialising a Lagrangian bundle (= no singularities), *i.e.* local coordinates $(\mathbf{a}, \boldsymbol{\alpha})$ on M such that locally

$$\omega = \sum_{k=1}^{n} \mathrm{d}a^k \wedge \mathrm{d}\alpha^k.$$

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Moreover $\mathbf{a}$ give rise to an atlas $\mathcal{A}$ of special coordinates on $B \rightsquigarrow$ affinely flat geometry.

Important corollaries

A topological consequence

The structure group of a Lagrangian bundle reduces to

$$\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) := \mathrm{GL}(n,\mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n.$$

$\Rightarrow$ topological classification (monodromy, Chern class).

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A subtler consequence

The atlas $\mathcal{A}$ on B makes it into an integral affine manifold, *i.e.* the changes of coordinates are constant on connected components and lie in

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) := \mathrm{GL}(n,\mathbb{Z}) \ltimes \mathbb{R}^n.$$

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- $\mathbb{R}^n$ with its standard atlas ✓;

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- Let $D : B \to B'$ be a local diffeomorphism. If B' is integral affine, D induces an integral affine structure on B;
- If Γ is a group acting by integral affine diffeomorphisms on $(B, \mathcal{A})$, then B/Γ inherits an integral affine structure from $(B, \mathcal{A}) \rightsquigarrow$ linear holonomy $\mathfrak{l} : \pi_1(B) \to \mathrm{GL}(n, \mathbb{Z})$;

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- Concretely, $\mathbb{T}^n$ ✓, S^2 x [Benzecri 1960, Milnor 1958], 3-dimensional lens spaces x [Smillie 1981] $\rightsquigarrow$ definition of radiance obstruction [Goldman and Hirsch 1984].

Invariants at a glance

Topological invariants

- Monodromy $\rho : \pi_1(B) \to \mathrm{GL}(n, \mathbb{Z})$

$$\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n / \mathbb{Z}^n$$

- Chern class $c \in \mathrm{H}^2(B; \mathbb{Z}^n_\rho)$

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Invariants of the base $(B, \mathcal{A})$

- Linear holonomy $\mathfrak{l} : \pi_1(B) \to \mathrm{GL}(n, \mathbb{Z})$

$$\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$$

- Radiance obstruction $r_{(B,\mathcal{A})} \in \mathrm{H}^1(B; \mathbb{R}^n_{\mathfrak{l}})$
- $\mathfrak{l}^{-T} = \rho$

Construction problem

Question (Dazord and Delzant, Zung)

Given $(B, \mathcal{A})$, which $\mathbb{T}^n$-fibre bundles over B are Lagrangian and induce $\mathcal{A}$?

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Remarks

- Restrict to $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$-bundles [Baier 2001];
- Require that monodromy $= \mathfrak{l}^{-T}$;
- Need to construct an appropriate symplectic form on $M \rightsquigarrow$ issue is closedness [Sansonetto and S. 2011] $\Rightarrow$ cohomological invariant.

Dazord-Delzant homomorphism

Theorem (Dazord-Delzant 1987)

Given $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$, isomorphism classes of Lagrangian bundles over $(B, \mathcal{A})$ is in $1-1$ correspondence with the kernel of a homomorphism

$$\mathcal{D}_{(B,\mathcal{A})} : \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}^{-T}}) \to \mathrm{H}^3(B; \mathbb{R}).$$

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Theorem (S. 2011)

Up to some isomorphisms,

$$\mathcal{D}_{(B,\mathcal{A})}(c) = c \cup r_{(B,\mathcal{A})},$$

where $r_{(B,\mathcal{A})}$ is the radiance obstruction of $(B, \mathcal{A})$ and $\cup$ denotes an appropriate cup product.

Local symplectic trivialisations

Let $\pi : (M, \omega) \to B$ be a Lagrangian bundle and let $U \subset B$ be a coordinate neighbourhood which trivialises π. Let $s_U : U \to M$ be a local Lagrangian section. Consider the map

$$\begin{aligned} \Upsilon_U : \mathrm{T}^* U &\to \pi^{-1}(U) \\ (b, \beta) &\mapsto g^1_\beta(s_U(b)), \end{aligned}$$

where g^1_β is the time-1 flow of the vector field v_β defined by

$$\pi^*(\beta) = \iota(v_\beta)\omega.$$

Period lattice bundle

The subbundle $P_U \subset \mathrm{T}^* U$ defined by

$$P_b := \{(b, \beta) \in \mathrm{T}^*_b B : \Upsilon_U(b, \beta) = s_U(b)\}$$

is a Lagrangian submanifold of $(\mathrm{T}^* U, \Omega_U)$.

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is a Lagrangian submanifold of $(\mathrm{T}^* U, \Omega_U)$. The map Υ_U descends to a fibre-preserving symplectomorphism

$$(\mathrm{T}^* U / P_U, \omega_{0,U}) \cong (\pi^{-1}(U), \omega).$$

$\rightsquigarrow$ existence of a reference bundle.

Symplectic reference bundles

Definition

Let $(B, \mathcal{A})$ be an integral affine manifold and define $P_{(B,\mathcal{A})} \subset \mathrm{T}^*B$ by

$$P_{(B,\mathcal{A}),U} = \{(b, \mathbf{p}) \in \mathrm{T}^*_b B : \mathbf{p} \in \mathbb{Z}\langle \mathrm{d}a^1_U, \ldots, \mathrm{d}a^n_U \rangle\}.$$

The bundle

$$(\mathrm{T}^*B / P_{(B,\mathcal{A})}, \omega_0) \to B$$

is the symplectic reference Lagrangian bundle of $(B, \mathcal{A})$.

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$\rightsquigarrow$ any Lagrangian bundle over B is locally isomorphic to a symplectic reference bundle.

Remarks

- Symplectic reference bundles contain information about the affinely flat geometry of $(B, \mathcal{A})$;
- The subbundle $P_{(B,\mathcal{A})}$ is called the period lattice bundle associated to $(B, \mathcal{A})$;
- Lagrangian bundles are assembled by symplectically gluing together local models of symplectic reference bundles;
- No equivalent when there are singularities [Vu Ngoc].

Almost Lagrangian bundles

Fix an integral affine manifold $(B, \mathcal{A})$ with linear holonomy $\mathfrak{l}$.

Definition

A $\mathbb{R}^n/\mathbb{Z}^n$-bundle over B with structure group $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ whose monodromy is $\mathfrak{l}^{-T}$ is called an almost Lagrangian bundle over $(B, \mathcal{A})$.

Some remarks

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- Almost Lagrangian bundles are topologically indistinguishable from Lagrangian bundles;
- Can use the affine structure $\mathcal{A}$ on B to construct local action-angle coordinates on M. Over each coordinate neighbourhood U, can construct a bundle isomorphism

$$\Upsilon_U : \pi^{-1}(U) \to \mathrm{T}^*U/P_{(B,\mathcal{A}),U},$$

where

$$P_{(B,\mathcal{A}),U} := \{(\mathbf{a}, \mathbf{p}) \in \mathrm{T}^*U : \mathbf{p} \in \mathbb{Z}\langle \mathrm{d}a^1, \ldots, \mathrm{d}a^n \rangle\},$$

$\rightsquigarrow$ have locally defined symplectic forms $\Phi_U^*\omega_{0,U}$. What about globally?

Transition functions

Let $\pi : M \to (B, \mathcal{A})$ be an almost Lagrangian bundle. Let U_i denote coordinate and trivialising neighbourhoods with action-angle coordinates $(\mathbf{a}_i, \boldsymbol{\alpha}_i)$. The transition functions Φ_{ji} are given by

$$\Phi_{ji}(\mathbf{a}_i, \boldsymbol{\alpha}_i) = (A_{ji}\mathbf{a}_i + d_{ji}, A_{ji}^{-T}\boldsymbol{\alpha}_i + \mathbf{g}_{ji}(\mathbf{a}_i)),$$

where

- $A_{ji} \in \mathrm{GL}(n, \mathbb{Z})$, $d_{ji} \in \mathbb{R}^n$ are constant;
- $\mathbf{g}_{ji}(\mathbf{a}_i)$ are smooth local functions (arising from no preferred choice of section).

Some observations

- The matrices A_{ji} give the monodromy of the bundle (fixed to be $\mathfrak{l}^{-T}$);
- The functions $\mathbf{g}_{ji}(\mathbf{a}_i)$ can be thought of as the Chern class c of the bundle ($\in \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}})$);

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- The functions $\mathbf{g}_{ji}(\mathbf{a}_i)$ can be thought of as the Chern class c of the bundle ($\in \mathrm{H}^2(B; \mathbb{Z}^n_{\mathfrak{l}})$);
- The bundle is Lagrangian if and only if

$$\tau_{ji} := \Phi^*_{ji}\omega_{0,j} - \omega_{0,i}$$

yields a globally defined exact 3-form [Dazord and Delzant].

Realisability theorem

Lemma

The symplectic form ω_0 on the symplectic reference bundle associated to $(B, \mathcal{A})$ defines a cohomology class

$$w_0 \in \mathrm{H}^1(B; \mathbb{R}^n_{\mathfrak{l}})$$

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Theorem (Dazord-Delzant 1987, S. 2011)

An almost Lagrangian bundle over $(B, \mathcal{A})$ with Chern class c is Lagrangian iff

$$\tau = c \cup w_0 = 0,$$

where $\tau \in \mathrm{H}^3(B; \mathbb{R})$ is the cohomology class defined by the Čech cocycles τ_{ji}.

Why should it be true?

Look at $\tau_{ji} = \Phi_{ji}^{*}\omega_{0,j} - \omega_{0,i}$. It is a locally defined 2-form which depends on the functions $\mathbf{g}_{ji} \rightsquigarrow$ the dependence on c. Also, τ_{ji} measures whether the locally defined forms $\omega_{0,i}$ patch together and these are local de Rham representatives of the cohomology class $w_0 \rightsquigarrow$ the dependence on w_0.

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Two methods of proof:

1. Using spectral sequence arguments;
2. Using ladders of short exact sequences of sheaves of sections of various bundles.

Radiance obstruction

Let $\mathcal{A} = \{(U_i, \varphi_i)\}$ be an integral affine atlas on B. Define affine trivialisations

$$\begin{aligned} \mathrm{T}^{\mathrm{Aff}} U_i &\to U_i \times \mathbb{R}^n \\ (p, \mathbf{v}) &\mapsto (p, D\varphi_i(p)(\mathbf{v}) + \varphi_i(p)) \end{aligned}$$

$\rightsquigarrow$ affine tangent bundle $\mathrm{T}^{\mathrm{Aff}} B \to B$.

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Lemma (Goldman and Hirsch 1980)

The obstruction to the existence of a flat (= constant) section of $\mathrm{T}^{\mathrm{Aff}} B \to B$ is the radiance obstruction $r_{(B,\mathcal{A})} \in \mathrm{H}^1(B; \mathbb{R}^n_{\mathfrak{l}})$.

An observation

The symplectic form ω_0 defines an isomorphism Ψ from the sheaf of flat sections of $\mathrm{T}^{\mathrm{Aff}}B \to B$ to the sheaf of 1-forms on the fibres of any almost Lagrangian bundle over $(B, \mathcal{A})$ defined by

$$s \mapsto \iota(s)\omega_0$$

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Lemma

If Ψ also denotes the induced isomorphism on cohomology groups with coefficients in the above sheaves, then

$$\Psi(r_{(B,\mathcal{A})}) = w_0.$$

Main theorem

Theorem (S. 2011)

An almost Lagrangian bundle over $(B, \mathcal{A})$ with Chern class c is Lagrangian if and only if

$$c \cup r_{(B,\mathcal{A})} = 0.$$

An example

Let $(B, \mathcal{A}) = (\mathbb{R}^2 \setminus \{\mathbf{0}\}, \mathcal{A}_1)^{\times 3}$, where $\mathcal{A}_1$ is defined by the representation

$$\begin{aligned} \mathfrak{a}_1 : \pi_1(\mathbb{R}^2 \setminus \{\mathbf{0}\}) \cong \mathbb{Z} &\to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2) \\ k &\mapsto \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

$(B, \mathcal{A})$ is radiant, so all cohomology classes in $\mathrm{H}^2(B; \mathbb{Z}^6_{(\mathfrak{l}_1^{-T})^3})$ can be realised.

Further work

- Use this result to study integral affine manifolds (*e.g.* the above result shows that there exist no closed radiant integral affine manifolds);
- Extend this result to isotropic (✓, but needs some care) and singular cases (work in progress);
- Explore meaning for CIHS (*e.g.* can the above examples be compactified to Lagrangian fibrations? What do the corresponding singularities look like?).

Thank you

Thank you for your attention.