Explicit Construction of K3 Dynamical Models and their Hamiltonian Monodromy

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Integrable Systems and Lagrangian Fibrations

Definition

A Hamiltonian system (M^{2n}, ω, H) is called completely integrable in the sense of Liouville, if there are n independent functions $f_1(= H), f_2, \ldots, f_n$ on M which Poisson commute.

Definition

A Lagrangian fibration of a symplectic manifold (M^{2n}, ω) is a mapping $f : M \to B^n$, such that there is an open dense subset $B_0 \subset B$ on whose preimage f is locally trivial and each fibre $f^{-1}(b)$ $(b \in B_0)$ is a (connected compact) Lagrangian submanifold of (M, ω).

Almost Toric Lagrangian Fibrations

Definition (Leung-Symington '10)

An almost toric Lagrangian fibration $f : (M, \omega) \to B$ is a Lagrangian fibration around whose critical points there is a Darboux coordinate system (p, q) with $\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i$ such that the projection $f = (f_1, \ldots, f_k, f_{k+1}, \ldots, f_n)$ is given as $f_i(p, q) = p_i$ for $i \leq k$ and as in one of the two forms $f_i(p, q) = p_i^2 + q_i^2$ or $(f_i, f_j)(p, q) = (p_i q_i + p_j q_j, p_i q_j - p_j q_i)$ for other components.

An almost toric Lagrangian fibration is a collection of local structures of completely integrable systems with compact fibres admitting only centre-centre and focus-focus equilibria (or their products).

Hamiltonian Monodromy

For a Lagrangian fibration $f : M \to B$, where $B_0 \subset B$ is an open dense subset over which f is locally trivial, we have $H_1(f^{-1}(b), \mathbb{Z}) \cong \mathbb{Z}^n$ for $b \in B_0$. The Hamiltonian monodromy can be described by the group representation

$$\pi_1(B_0, b_0) \to \mathrm{Aut}(H_1(f^{-1}(b_0), \mathbb{Z})) \cong SL(n, \mathbb{Z}),$$

where $b_0 \in B_0$ is the reference point.

K3 Surfaces

A compact complex surface M is called K3, if it is simply connected and if its canonical bundle K_M is trivial, *i.e.* if there is a non-vanishing holomorphic two form on M.

Example. Kummer Surface:
Take a complex torus $T = \mathbb{C}^2/\Lambda$, where $\Lambda \cong \mathbb{Z}^4$. The involution $\mathbb{C}^2 \ni (x, y) \mapsto (-x, -y) \in \mathbb{C}^2$ induces the one ι of T. The quotient surface $T/\langle\iota\rangle$ has 16 singular points. After the minimal desingularization of these singularities, we have a K3 surface, which is usually called a Kummer surface.

K3 Surfaces as Phase Spaces

Let M be a K3 surface and Ω a non-vanishing holomorphic two form on M.

Proposition

The differential two form $\omega := \mathrm{Re}\,(\Omega) = \left(\Omega + \overline{\Omega}\right)/2$ *is a real symplectic form over* M *which is regarded as a real four-dimensional manifold.*

For any function h on (an open subset of) M, we can consider the (real) Hamiltonian vector field X_h^ω with respect to ω: $\iota_{X_h^\omega}\omega = -\mathrm{d}h$.

Remark. The non-vanishing holomorphic two form is itself a holomorphic symplectic form on M. Thus, we can also consider the complex Hamiltonian vector field X_h^Ω of h with respect to Ω: $\iota_{X_h^\Omega}\Omega = -\mathrm{d}h$.

Hamilton Equation of a Holomorphic Function

Let M be a K3 surface and Ω a non-vanishing holomorphic two form, as before. Set $\omega = \mathrm{Re}(\Omega)$.

Proposition (Cf. Bates-Cushman '05)

If h is a holomorphic function over an open set of M, then

$$X^{\omega}_{\mathrm{Re}(h)} = \mathrm{Re}(X^{\Omega}_h),\ X^{\omega}_{\mathrm{Im}(h)} = \mathrm{Im}(X^{\Omega}_h).$$

Proposition (Cf. Bates-Cushman '05)

If h is a holomorphic function over an open set of M, then $\mathrm{Re}(h)$ and $\mathrm{Im}(h)$ Poisson commute with respect to the Poisson structure associated to ω.

Elliptic K3 Surfaces in Weierstraß Normal Form 1

We introduce the Weierstraß normal form of elliptic surfaces, in order to describe integrable systems over a K3 surface M.

Let $L \to B$ be a line bundle over a Riemann surface B and consider the vector bundle $E := L^{\otimes 2} \oplus L^{\otimes 3} \oplus \mathcal{O} \to B$. We take the $P_2(\mathbb{C})$-bundle $P := P(E) \to B$, whose homogeneous fibre coordinates are denoted by $(x : y : z)$. Choosing holomorphic sections $g_2 \in H^0(B, L^{\otimes 4})$ and $g_3 \in H^0(B, L^{\otimes 6})$, we consider the hypersurface $W \subset P$ defined by the equation

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3.$$

Elliptic K3 Surfaces in Weierstraß Normal Form 2

Restricting the projection $P \to B$ to W, we have an elliptic surface $\pi : W \to B$. W is called in Weierstraß normal form.

Since the total space W generally has singularity, we have to give the desingularization $\widehat{W} \to W$.

Now, we consider the case where $B = P_1(\mathbb{C})$ and $L = \mathcal{O}_{P_1(\mathbb{C})}(2)$. We denote the homogeneous coordinates of $B = P_1(\mathbb{C})$ by $(t_0 : t_1)$. On the open set where $t_1 \neq 0$, $y \neq 0$, and $z \neq 0$, we set

$$\Omega = \frac{\mathrm{d}\overline{x} \wedge \mathrm{d}t}{\overline{y}},$$

where $\overline{x} = x/z$, $\overline{y} = y/z$, and $t = t_0/t_1$.

Elliptic K3 Surfaces in Weierstraß Normal Form 3

Proposition

The holomorphic two form Ω can be extended over the open set $W^{\mathrm{reg}} \subset W$ of smooth points. Moreover, it gives rise to a non-vanishing holomorphic two form over $\widehat{W}$.

The smooth surface $\widehat{W}$ is an elliptic K3 surface.

Remark 1. The classification of singular fibres for elliptic surfaces is given by Kodaira, together with the conjugacy of monodromy matrices.

Remark 2. The integrability of any elliptic K3 surface, as a two-dimensional meromorphic Hamiltonian system, has been already known from the work of Markushevich.

Complex Hamilton Equation 1

For the elliptic K3 surface $\pi : \widehat{W} \to P_1(\mathbb{C})$, the local expression of complex Hamilton equation can be given as follows: On the open set where $t_1 \neq 0$ and $z \neq 0$, the complex Hamilton equaiton with respect to the Hamiltonian $t = t_0/t_1$ is

$$\begin{aligned} \dot{\overline{x}} &= -\overline{y} \\ \dot{\overline{y}} &= -\frac{12\overline{x}^2 - g_2}{2}. \end{aligned}$$

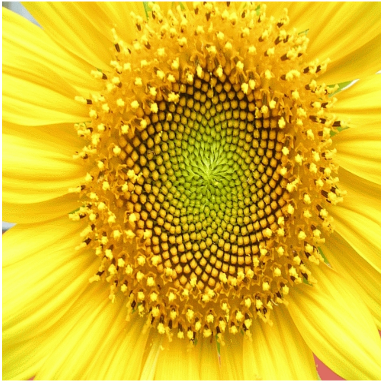
Complex Hamilton Equation 2

Recall the famous formula for the Weierstraß $\wp$-function:

$$\ddot{\wp} = \frac{12\wp^2 - g_2}{2}.$$

Since the real Hamiltonian vector fields $X^{\omega}_{\mathrm{Re}t}$ and $X^{\omega}_{\mathrm{Im}t}$ are exactly the real and the imaginary parts of the above complex Hamiltonian vector field, the dynamical systems can be solved essentially by the Weierstraß $\wp$-function.

Lattice Defect for the Sunflower



This picture indicates the existence of an integrable system with a monodromy matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ corresponding to an arc around several non-trivial singular loci.

Example 1

We describe an example of completely integrable systems on K3 surfaces.

We first consider the following rational elliptic surface.
Let $B = P_1(\mathbb{C}) : (t_0 : t_1)$ as before and choose the sections $g_2 \in H^0(B, \mathcal{O}_B(4))$ and $g_3 \in H^0(B, \mathcal{O}_B(6))$ as $g_2(t) = t^2$ and $g_3(t) = t^3$, where $t = t_0/t_1$. We denote the associated Weierstraß normal form by $\pi_W : W \to P_1(\mathbb{C})$. The discriminant D is described as $D(t) = g_2^3 - 27g_3^2 = -26t^6$.

Example 2

After desingularizing two singular points of type D_4 on W, we have a smooth rational elliptic surface $\pi_{\widehat{W}} : \widehat{W} \to P_1(\mathbb{C})$. The singular fibres of $\pi_{\widehat{W}}$ are lying over $t = 0$ and $t = \infty$ and of type I_0^* in Kodaira's notation, whose monodromy matrix is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Example 3

Next, we perturb the above elliptic surface by changing the sections as $g_2'(t) = (t - a_1)(t - a_2)(t - a_3)(t - a_4)$ and $g_3'(t) = (t - b_1)(t - b_2)(t - b_3)(t - b_4)(t - b_5)(t - b_6)$.
Here, a_i $(i = 1, \ldots, 4)$ and b_j $(j = 1, \ldots, 6)$ are complex numbers such that
$0 < |a_1|, |a_2|, |b_1|, |b_2|, |b_3| \ll 1 \ll |a_3|, |a_4|, |b_4|, |b_5|, |b_6|$.
We denote the perturbed elliptic surface by
$\pi_{W'} : W' \to P_1(\mathbb{C})$.

Since the discriminant $D'(t) = g_2'(t)^3 - 27 g_3'(t)^2$ has only simple roots, the perturbed fibration $\pi_{W'}$ has 12 singular fibres of type I_1 in Kodaira's notation.
(Six near $t = 0$ and six near $t = \infty$.)
Note that W' is a smooth rational surface.

Example 4

Now, we choose a complex number α with $1 < |\alpha| \ll |a_3|, |a_4|, |b_4|, |b_5|, |b_6|$ and take the double covering $\phi : \widetilde{B} \to B$ branched at $t = 1$ and $t = \alpha$. Pulling back the fibration $\pi_{W'} : W' \to B$ to $\widetilde{B}$ through ϕ, we obtain an elliptic K3 surface $\pi_{\widetilde{W'}} : \widetilde{W'} \to \widetilde{B}$.

The Weierstraß normal form for $\pi_{\widetilde{W'}}$ is given by
$\widetilde{g_2'}(T) = g_2'\left(\frac{1-\alpha T^2}{1-T^2}\right) \in H^0(\widetilde{B}, \mathcal{O}(8))$,
$\widetilde{g_3'}(T) = g_3'\left(\frac{1-\alpha T^2}{1-T^2}\right) \in H^0(\widetilde{B}, \mathcal{O}(12))$.

Example 5

The elliptic K3 surface $\pi_{\widetilde{W'}} : \widetilde{W'} \to \widetilde{B}$ has 24 singular fibres of type I_1.
The monodromy matrix corresponding to the closed path around the six singular loci near $T = 1/\sqrt{\alpha}$ and the six near $T = -1/\sqrt{\alpha}$ can be calculated as

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 6

Theorem

The elliptic K3 surface $\pi_{\widetilde{W'}} : \widetilde{W'} \rightarrow \widetilde{B}$ *is an almost toric Lagrangian fibration, whose equilibria are all focus-focus. There is an element* $\sigma \in \pi_1(\widetilde{B} \setminus \textit{singular loci})$ *which gives rise to the monodromy matrix*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The element σ *can be represented by a closed path around 12 singular loci.*

Vielen Dank!

Thank you very much!