From Hamiltonian monodromy and lattice defects to spiral phyllotaxis.

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FDIS 2011

Jena, Germany,

26-29 July 2011

My favorite subject - *energy levels* of simple finite particle *quantum* systems.

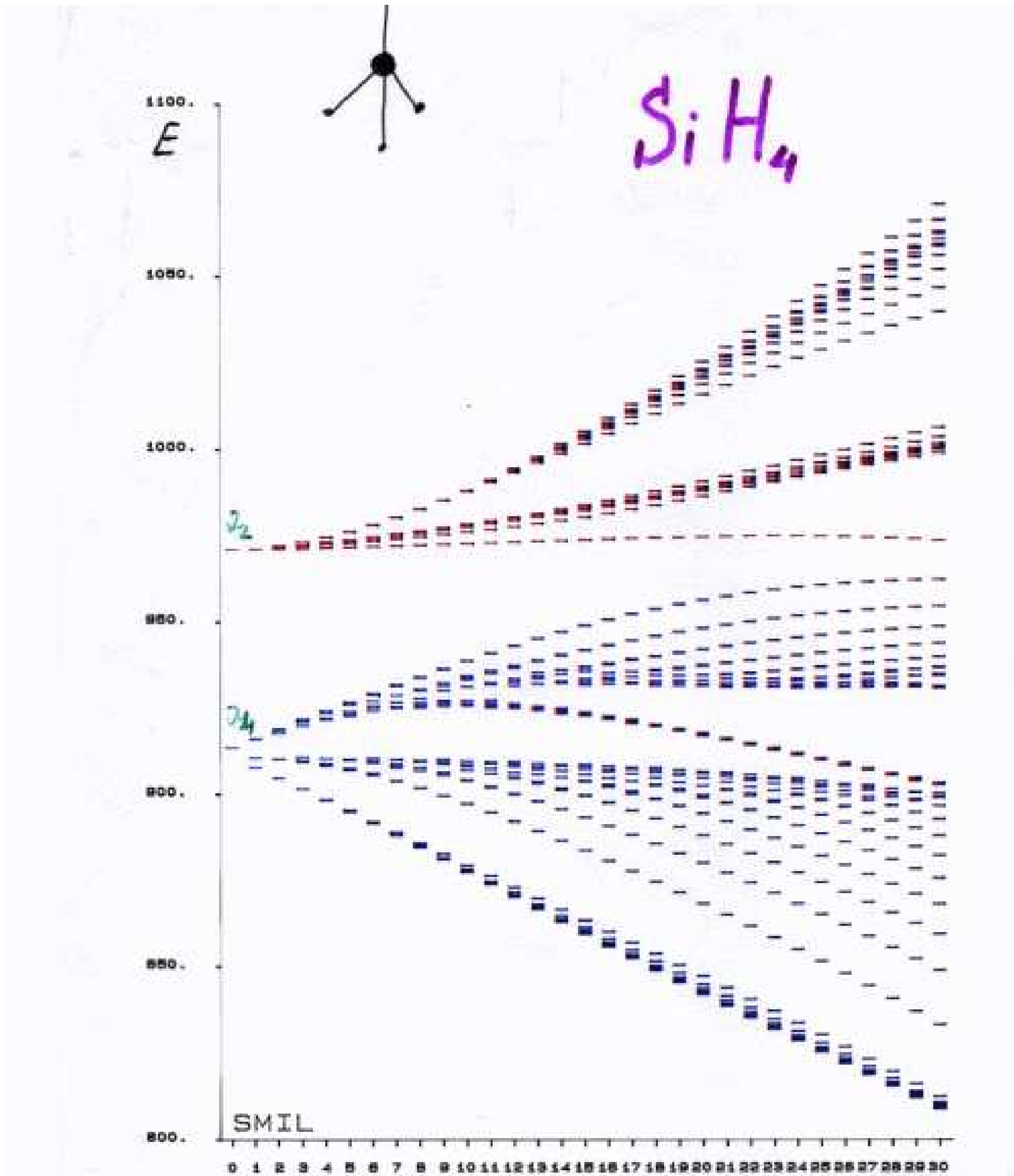
Typically they are classified by values of exact or approximate integrals of motion (energy, angular momentum, number of vibrational quanta, ...)

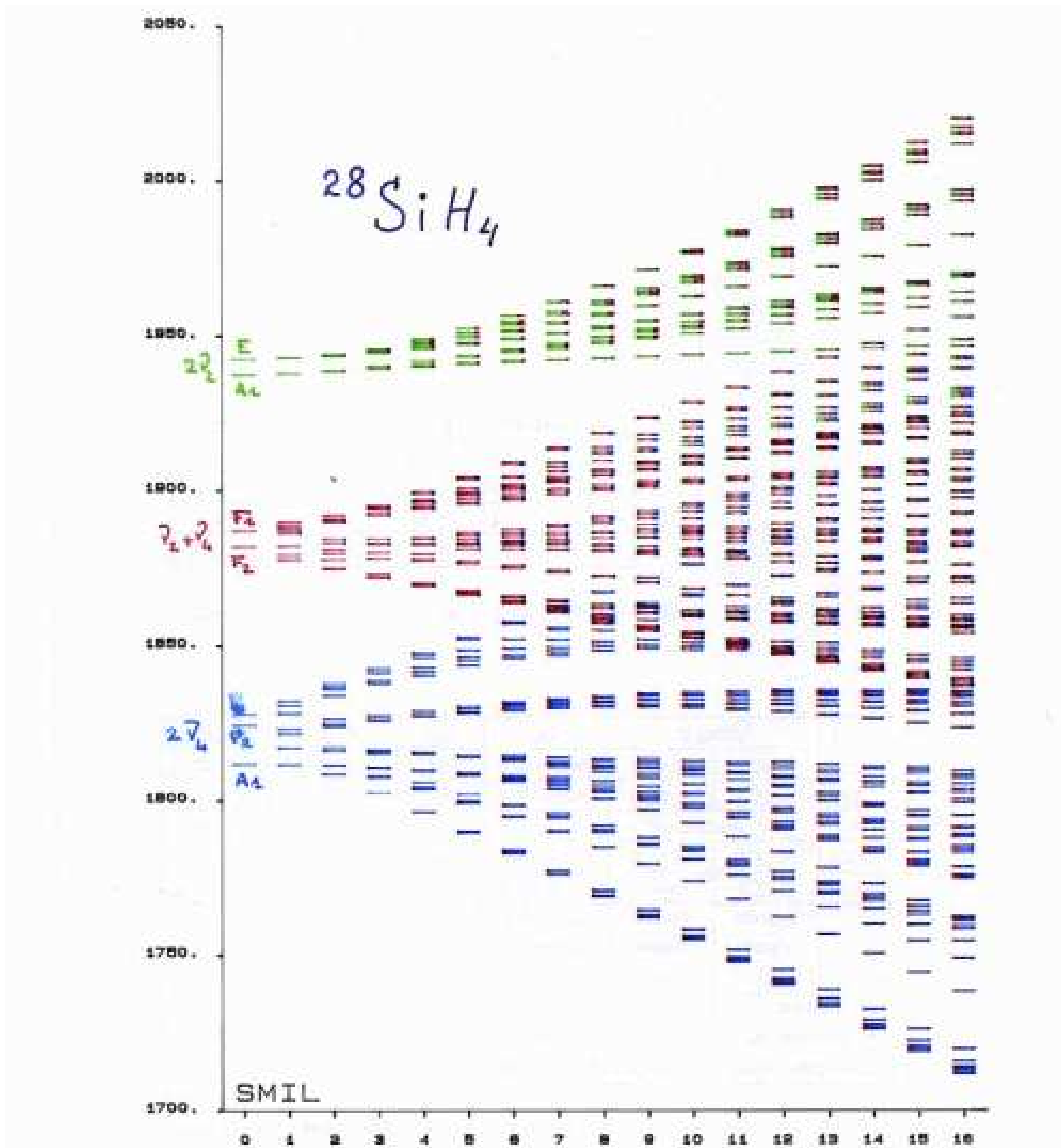
From the **classical mechanics** point of view we have **integrable** systems but the integrals can take only some **disrete** values.

From the **quantum mechanics** point of view each quantum state is an eigenfunction of a number of mutually commuting quantum operators.

Joint spectrum of commuting operators **form a pattern** which is locally a regular Z^N lattice, but globally it has many nontrivial features.

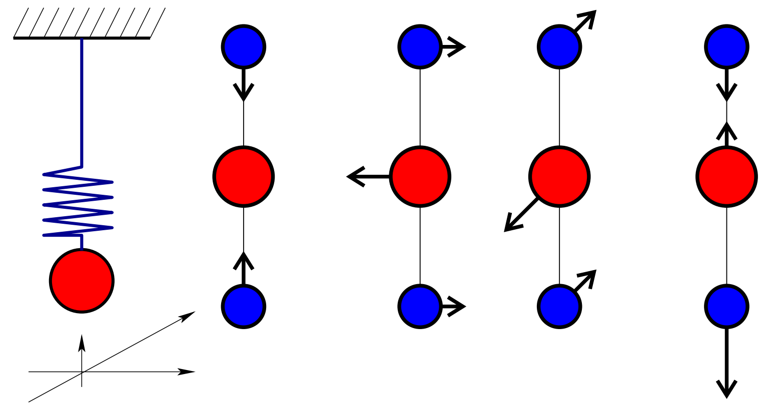
To describe some **universal features** of these **patterns** is the main topic of my work and of my today presentation.


SiH4
E
1100.
1050.
1000.
950.
900.
850.
800.
SMIL
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30



$^{28}SiH_4$
2050.
2000.
1950.
1900.
1850.
1800.
1750.
1700.
SMIL
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Monodromy of *swing-spring* with 1:1:2 resonance - model of Fermi resonance in CO_2



3D of freedom dynamical system - three resonant nonlinear oscillators - in the presence of axial symmetry.

CO_2 has four vibrational modes: symmetric and antisymmetric stretch and doubly degenerate bending. Antisymmetric vibration is out of resonance and can be “neglected“ (averaged).

Integrable model

$$L = \frac{1}{2}(z_2\bar{z}_3 - \bar{z}_2 z_3)\mathrm{i}, \tag{1}$$

$$N = \bar{z}_1 z_1 + \frac{1}{2}\bar{z}_2 z_2 + \frac{1}{2}\bar{z}_3 z_3, \tag{2}$$

$$H = aS + bR + cR^2 + E(N, L). \tag{3}$$

written in terms of invariant polynomials

$$R = \tfrac{1}{2}\bar{z}_2 z_2 + \tfrac{1}{2}\bar{z}_3 z_3 = (n_2 + n_3), \tag{4}$$

$$S = \tfrac{1}{4}(\bar{z}_1 z_3^2 + z_1\bar{z}_3^2 + z_1\bar{z}_2^2 + \bar{z}_1 z_2^2), \tag{5}$$

$$T = \tfrac{1}{4}(\bar{z}_1 z_3^2 - z_1\bar{z}_3^2 - z_1\bar{z}_2^2 + \bar{z}_1 z_2^2)\mathrm{i}, \tag{6}$$

with $z = q - ip, \bar{z} = q + ip, \{z, \bar{z}\} = 2i$

Qualitative models:

$N-$degree of freedom reduced systems

Integrable Hamiltonian models

Energy-momentum map images (singular and regular parts)

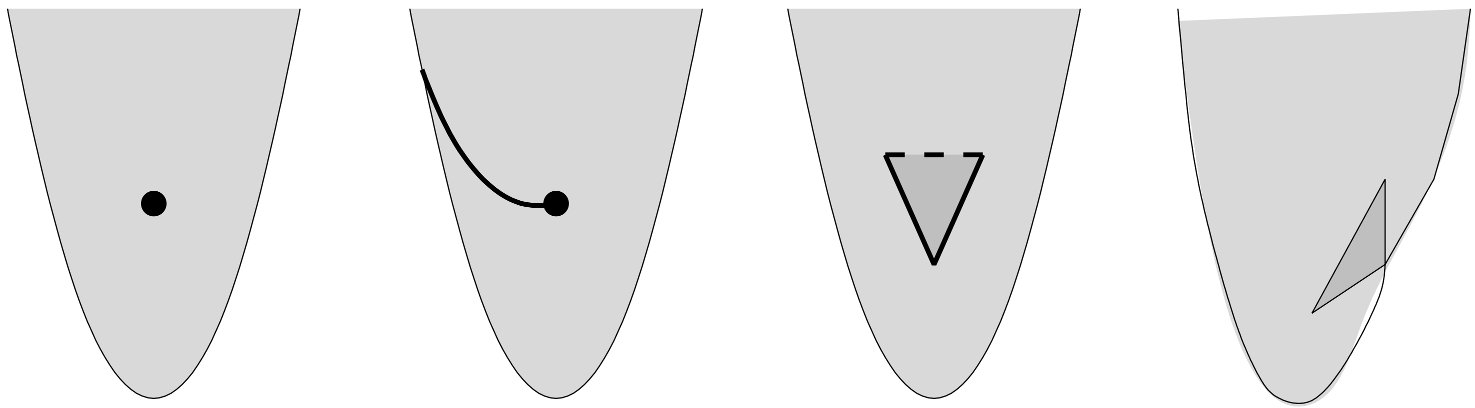
Toric fibrations with singularities.

Cushman R., Bates L. *Global aspects of classical integrable systems.* Birkhäuser, Basel, 1997

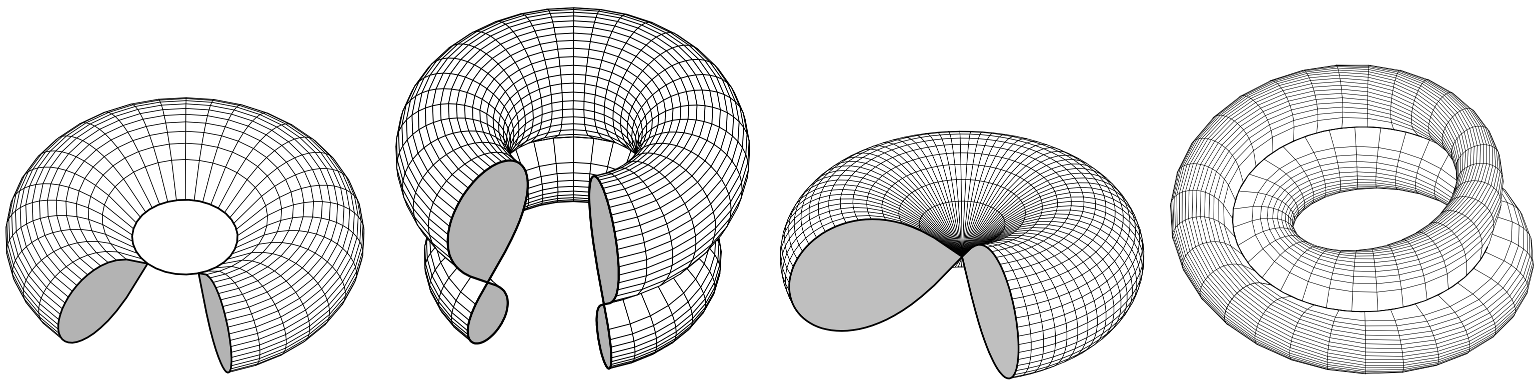
Duistermaat J.J. *On global action angle coordinates.* Comm. Pure Appl. Math. **33**, 687-706 (1980)

Nekhoroshev N.N. *Action-angle variables and their generalizations.* Trans. Moscow Math. Soc. **26**, 180-198 (1972)

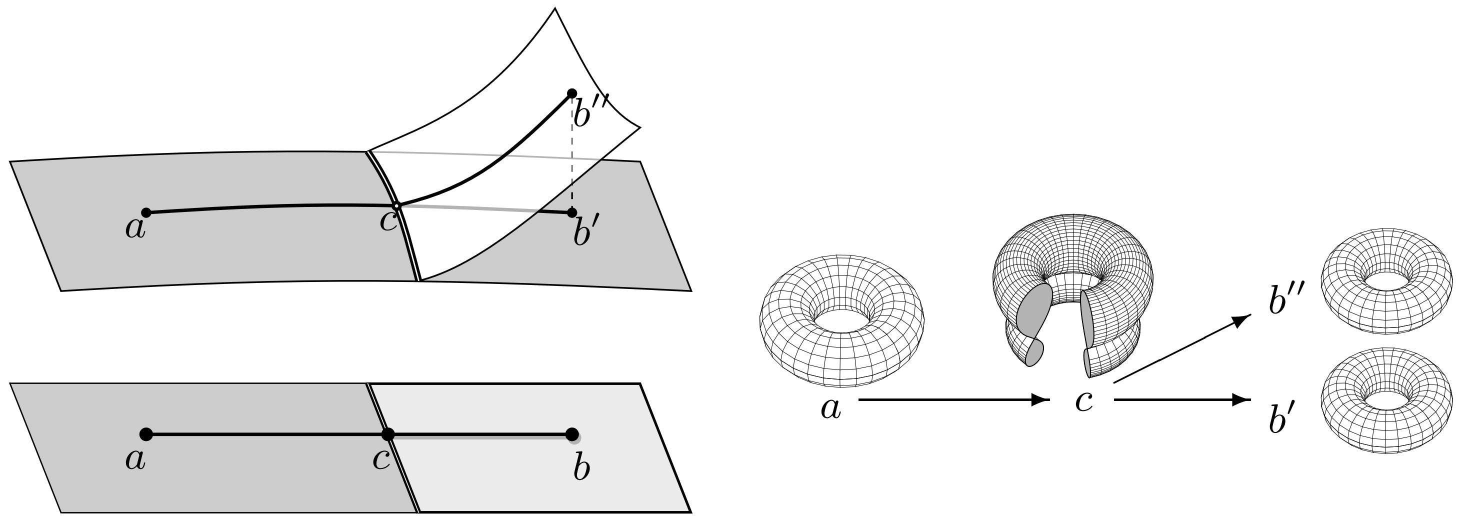
Bolsinov A.V., Fomenko A.T. *Integrable Hamiltonian Systems.* Chapman& Hall, 2004



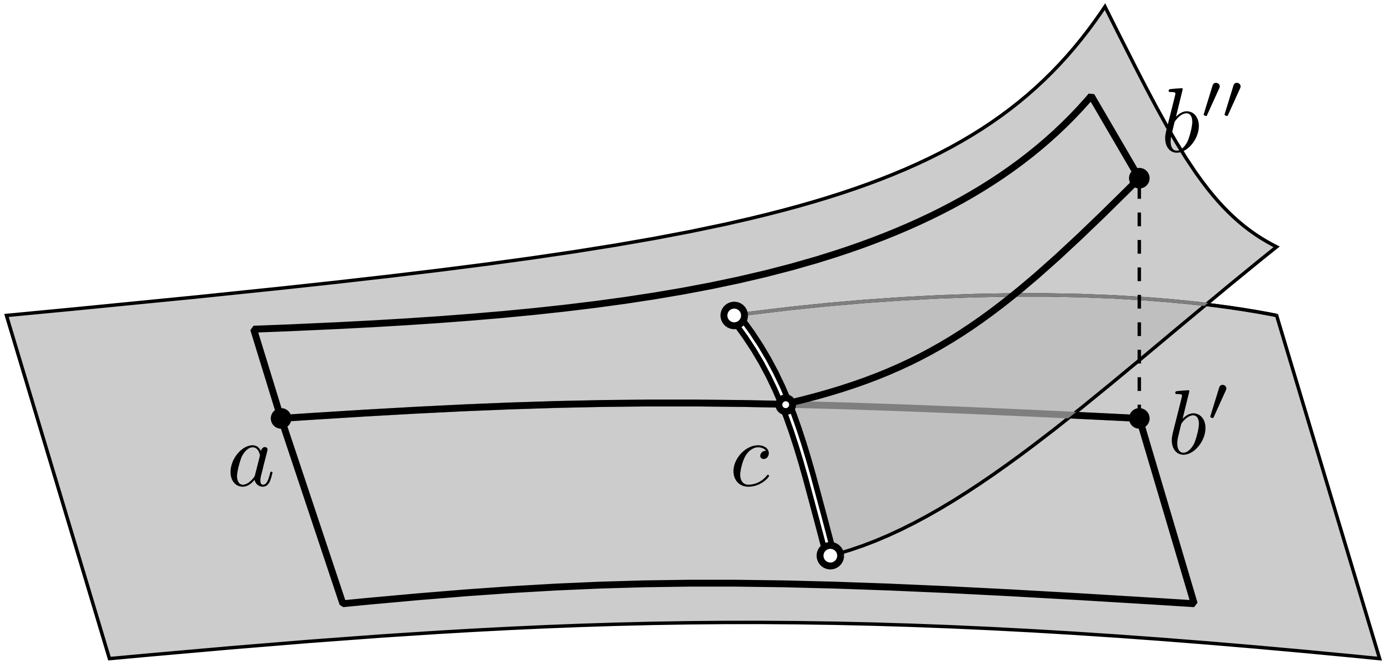
Typical images of the energy momentum map for completely integrable Hamiltonian systems with two degree of freedom in the case of integer monodromy, fractional monodromy, nonlocal monodromy, and bidromy. Values in light shaded area lift to single 2-tori; values in dark shaded area lift to two 2-tori.



Two dimensional singular fibers in the case of integrable Hamiltonian systems with two degrees of freedom (left to right): singular torus, bitorus, pinched and curled tori.



Example of overlapping lower cells in the 2D-image of the energy-momentum map: a multi-sheet cell unfolding surface (top left) and the corresponding image (bottom). Points a, b', b'', and c lift each to one connected component of the integrable fibration shown right; b' and b'' correspond to the same value b.



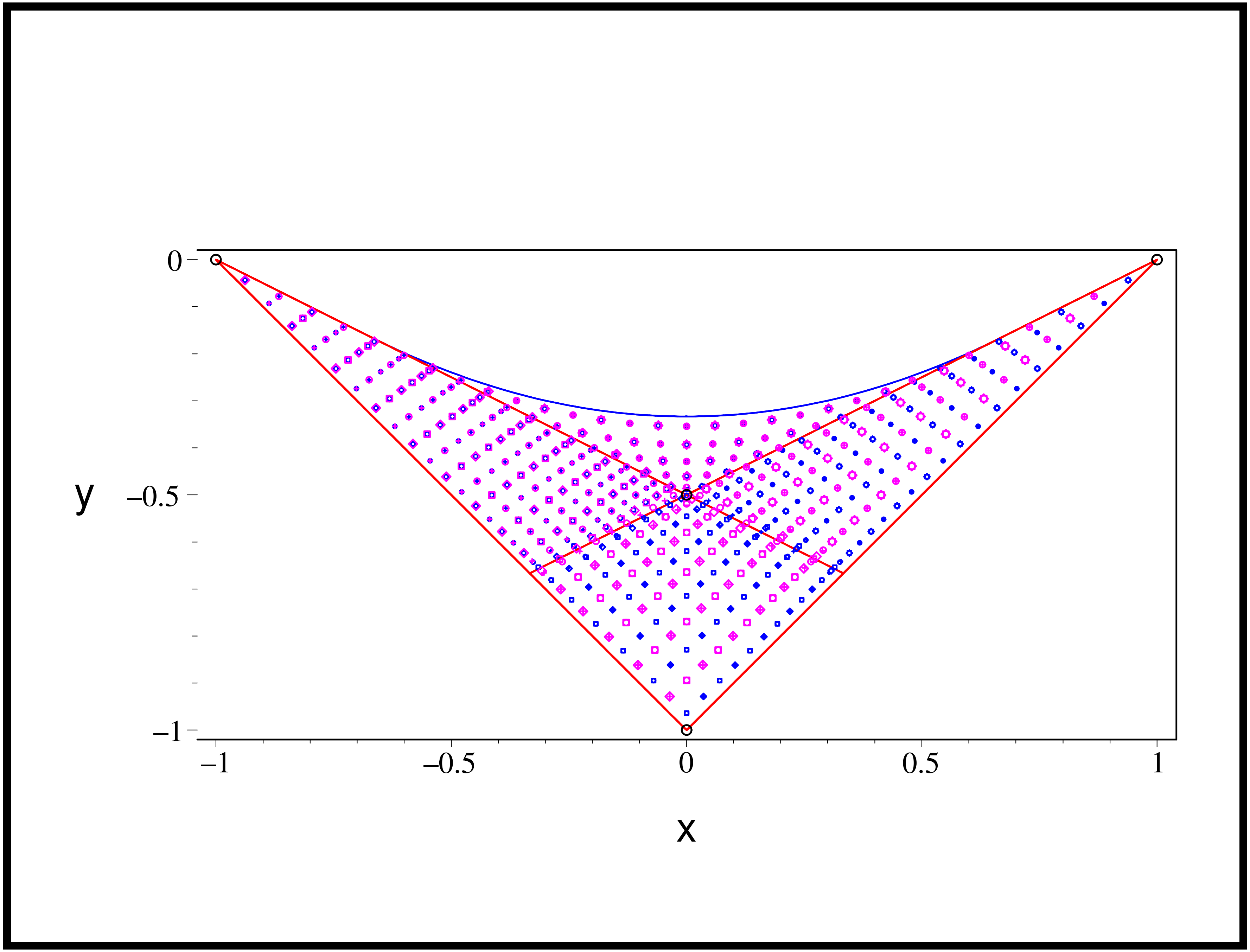
A single-sheet cell unfolding surface for the image of an energy momentum map with one self-overlapping lower cell: the inverse map can have one (point a) or two (points b' and b'') connected components.

Manakov top

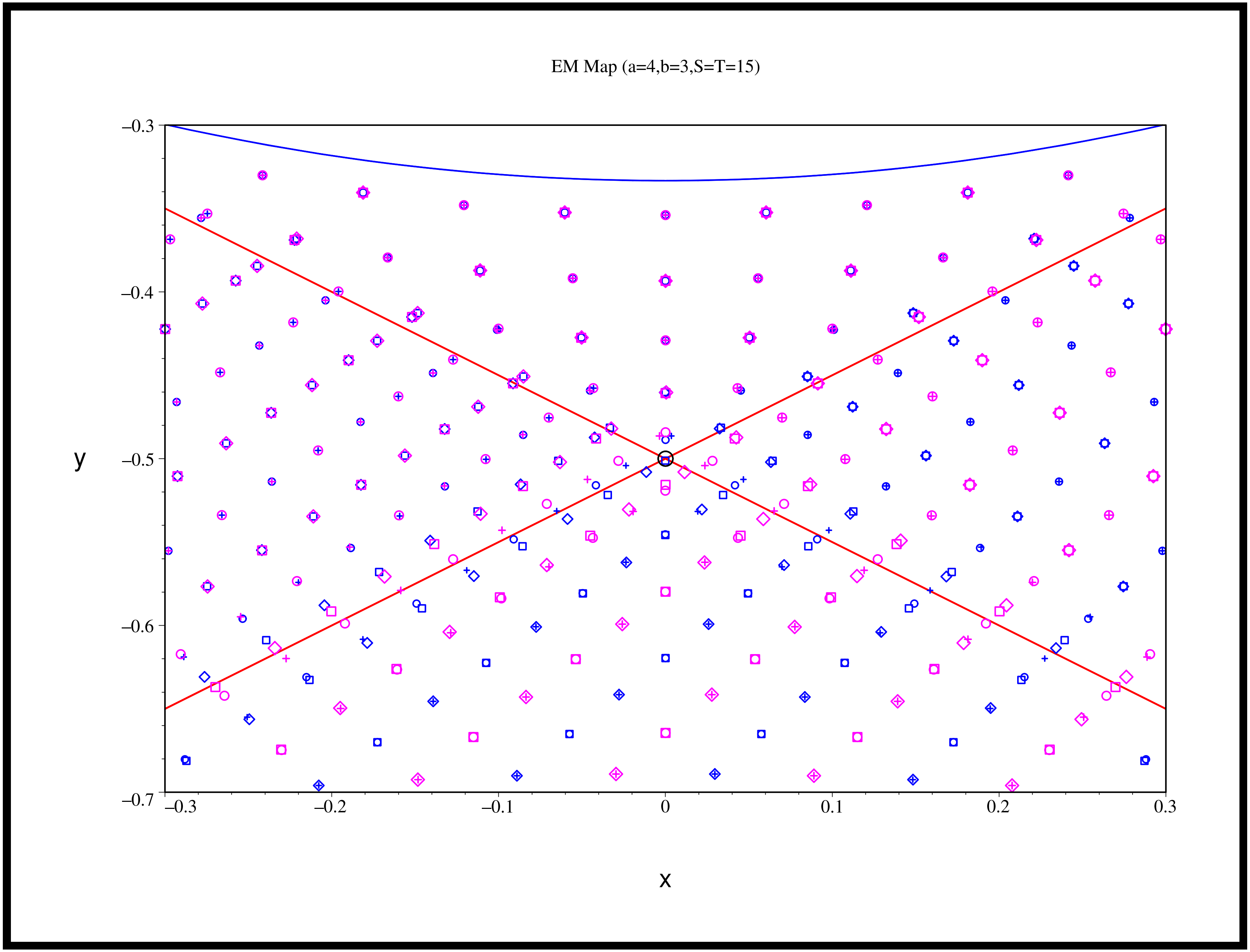
$$\begin{aligned} X &= s_1 t_1 + \frac{a-b-1}{1-a-b} s_2 t_2 + \frac{b-a-1}{1-a-b} s_3 t_3, \\ Y &= b(1-a)(s_2^2 + t_2^2) + 2b(1-a)\frac{b-a-1}{1-a-b} s_2 t_2 \\ &+ a(1-b)(s_3^2 + t_3^2) + 2a(1-b)\frac{a-b-1}{1-a-b} s_3 t_3 \end{aligned} \tag{7}$$

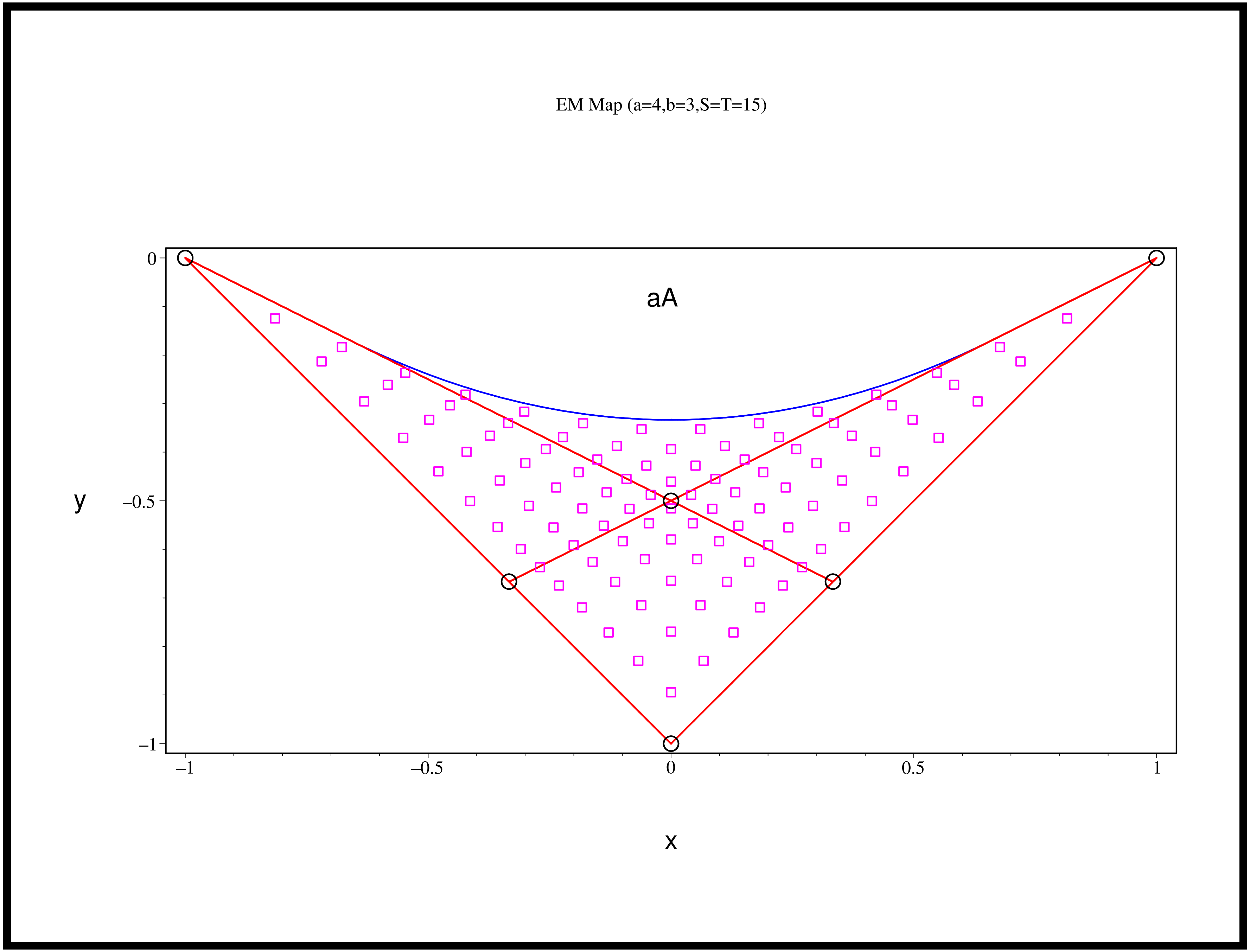
Here the generators s_i, t_i, $i = 1, 2, 3$, obey the standard commutation relations ($o(4) \simeq su_s(2) \otimes su_t(2)$):

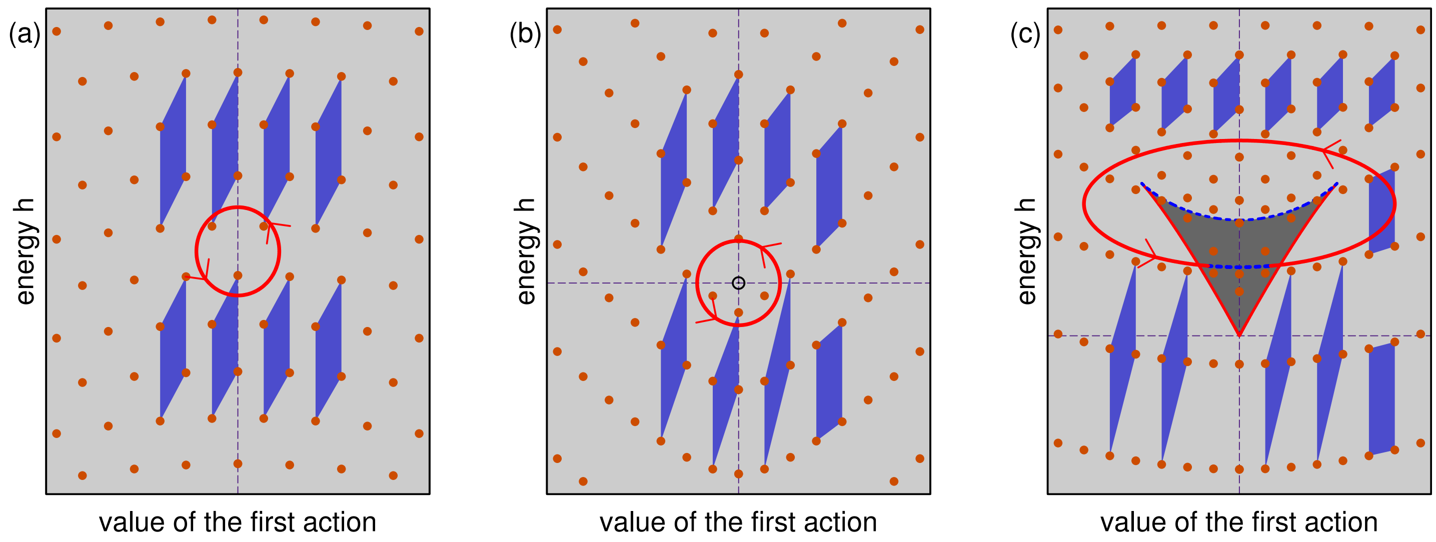
$$[s_i, s_j] = i\varepsilon_{ijk} s_k, \qquad [t_i, t_j] = i\varepsilon_{ijk} t_k, \qquad [s_i, t_j] = 0. \tag{8}$$



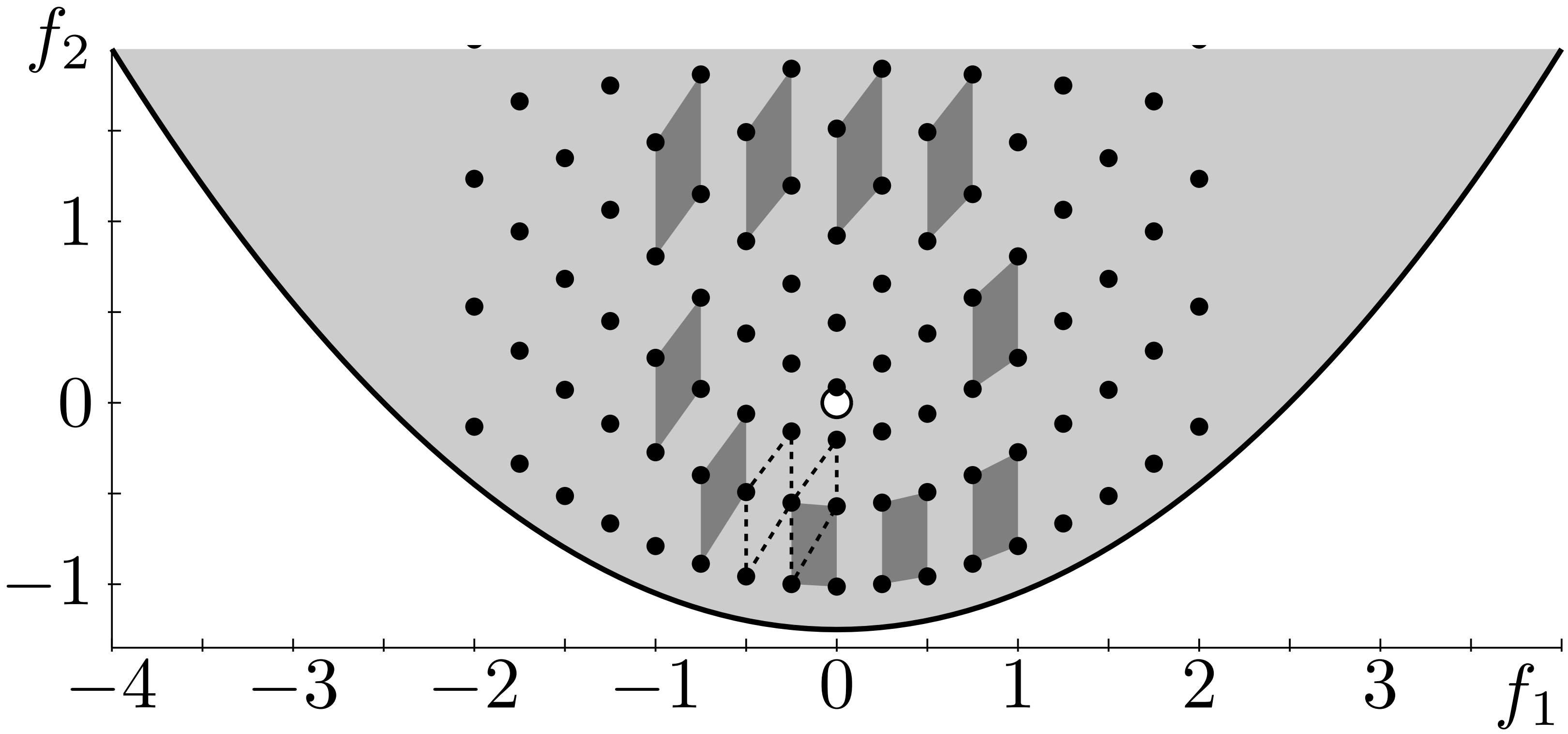
0
–0.5
–1
y
–1
–0.5
0
0.5
1
x


EM Map (a=4,b=3,S=T=15)
y
x
−0.3
−0.4
−0.5
−0.6
−0.7
−0.3
−0.2
−0.1
0
0.1
0.2
0.3

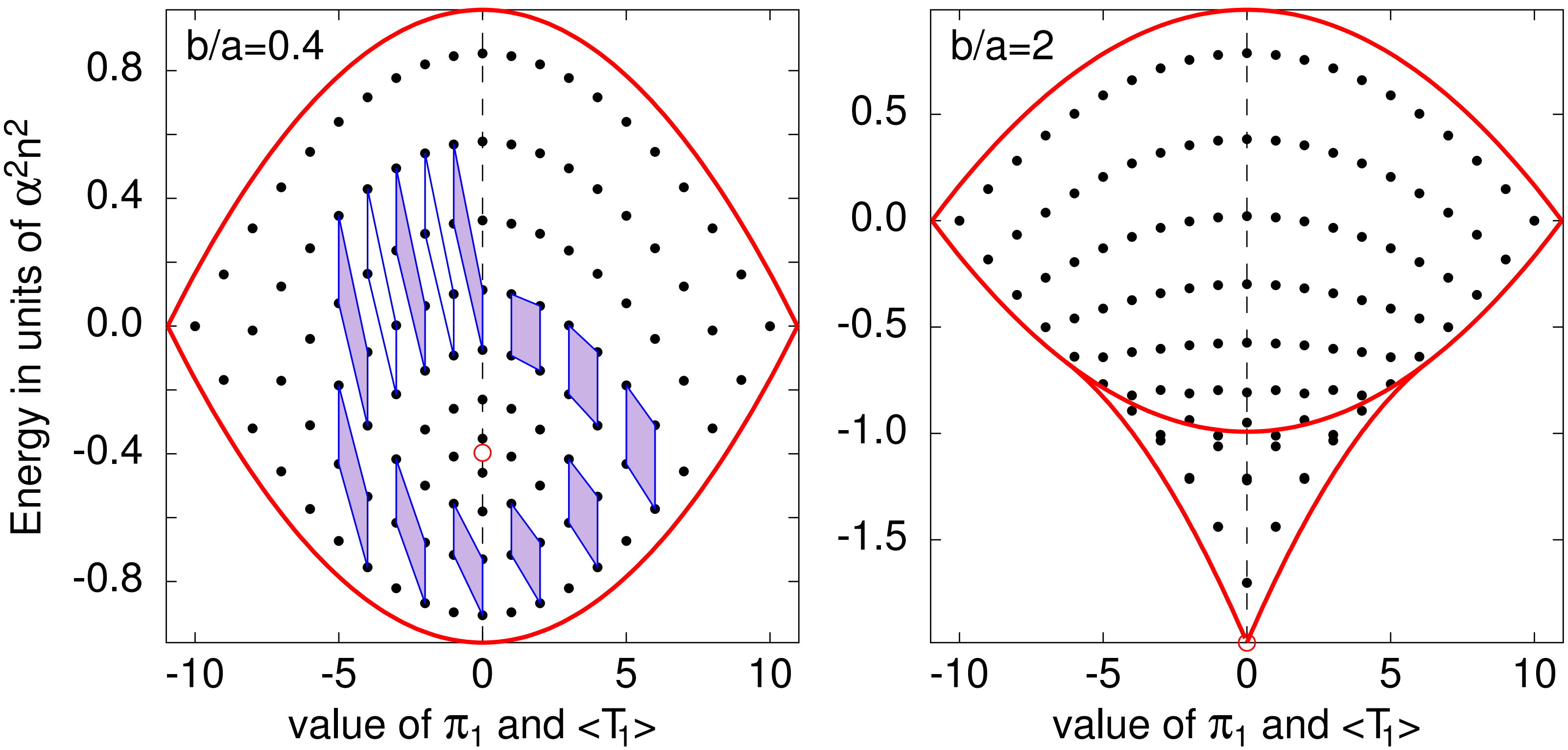

EM Map (a=4,b=3,S=T=15)
aA
0
−0.5
−1
y
−1
−0.5
0
0.5
1
x



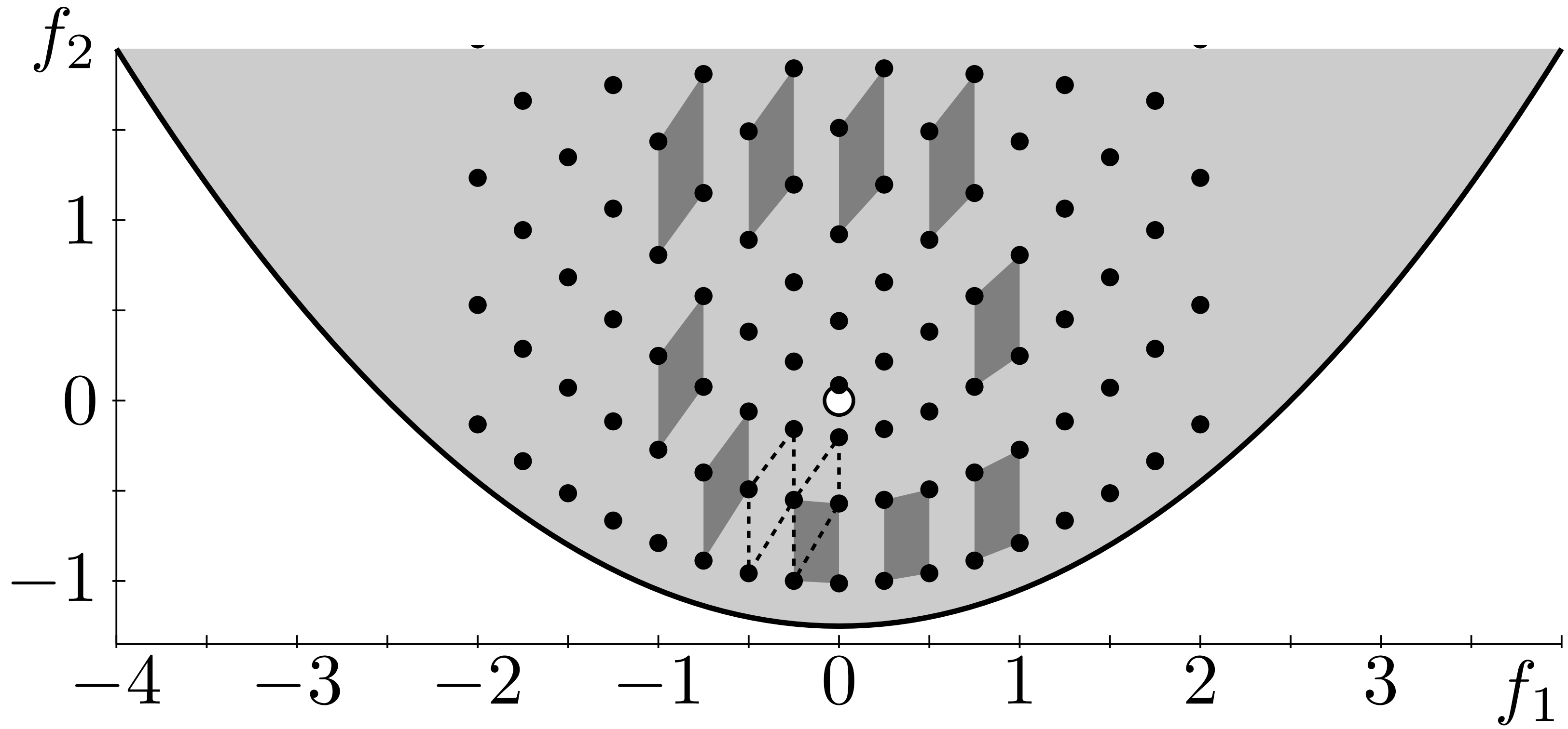
Quantum joint spectra for typical regions of the image of energy - momentum map for integrable problems.



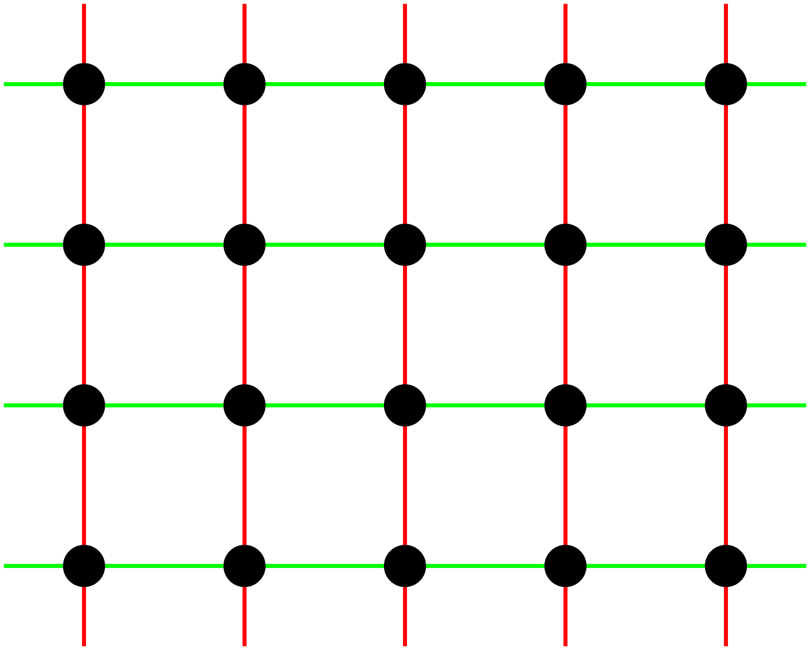
Quantum monodromy for $1 : (-1)$ resonant oscillator system.



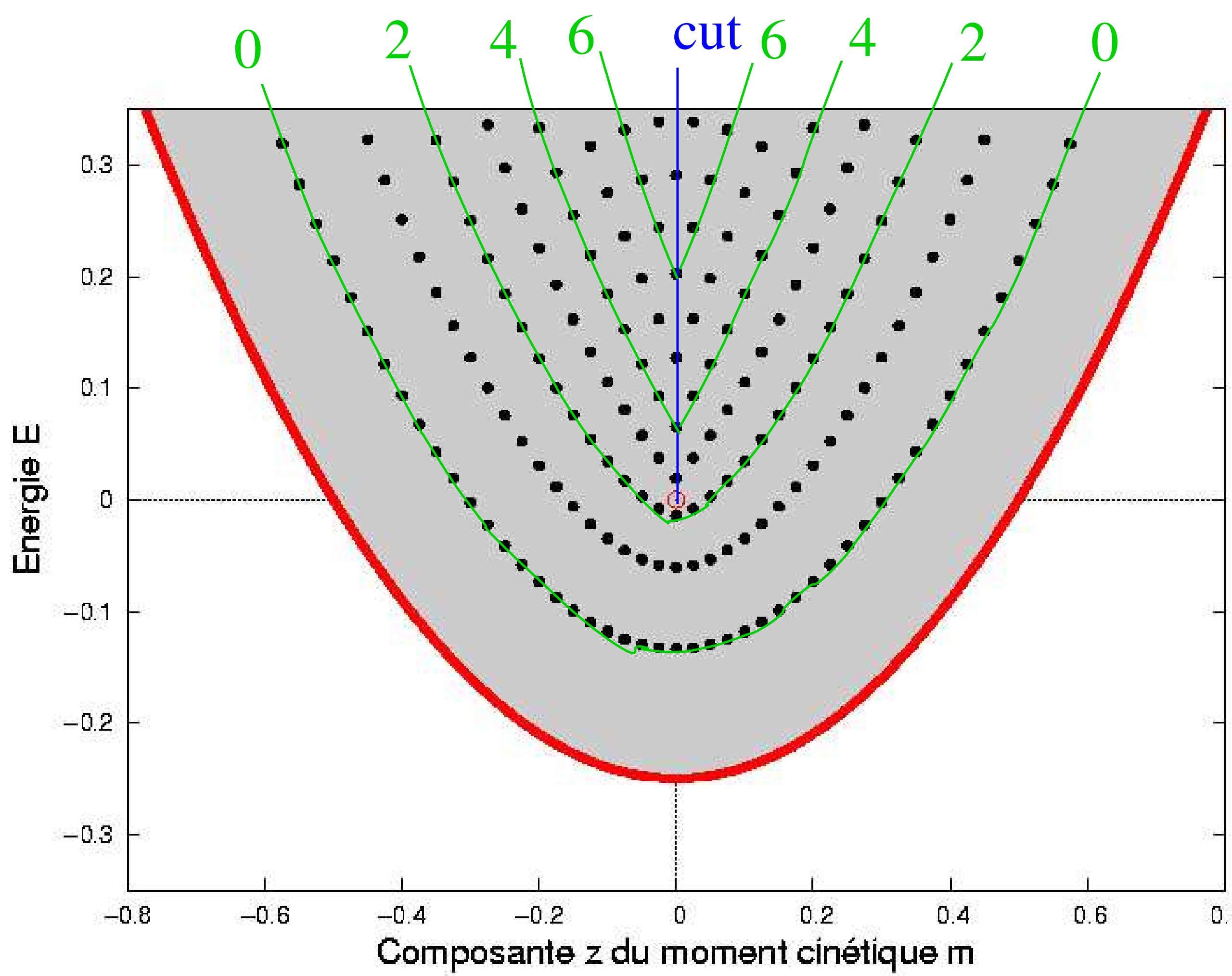
Hydrogen atom in orthogonal electric and magnetic fields.



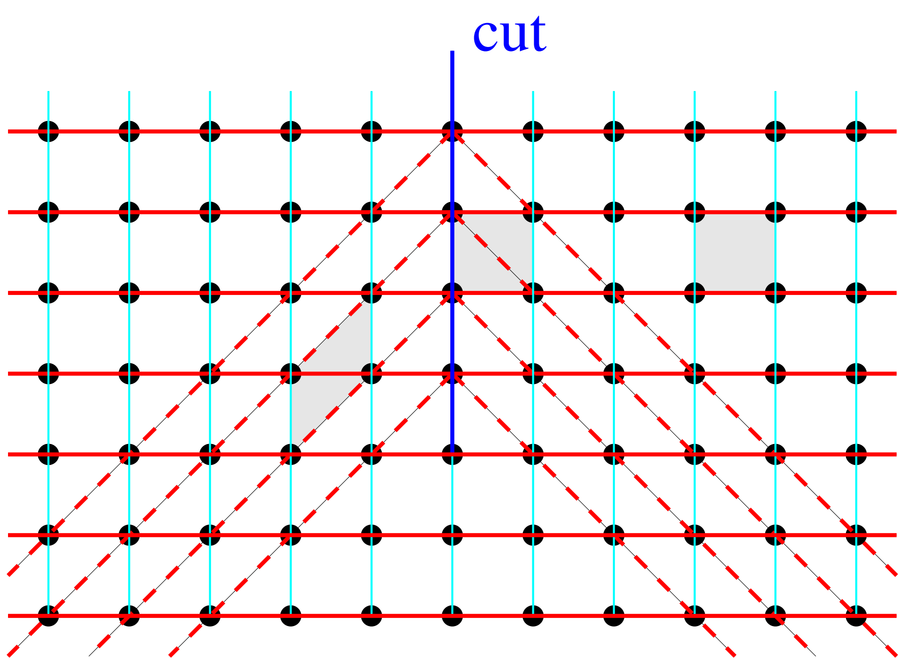
Quantum monodromy for $1 : (-1)$ resonant oscillator system.



Local representation of the regular part of the joint spectrum.

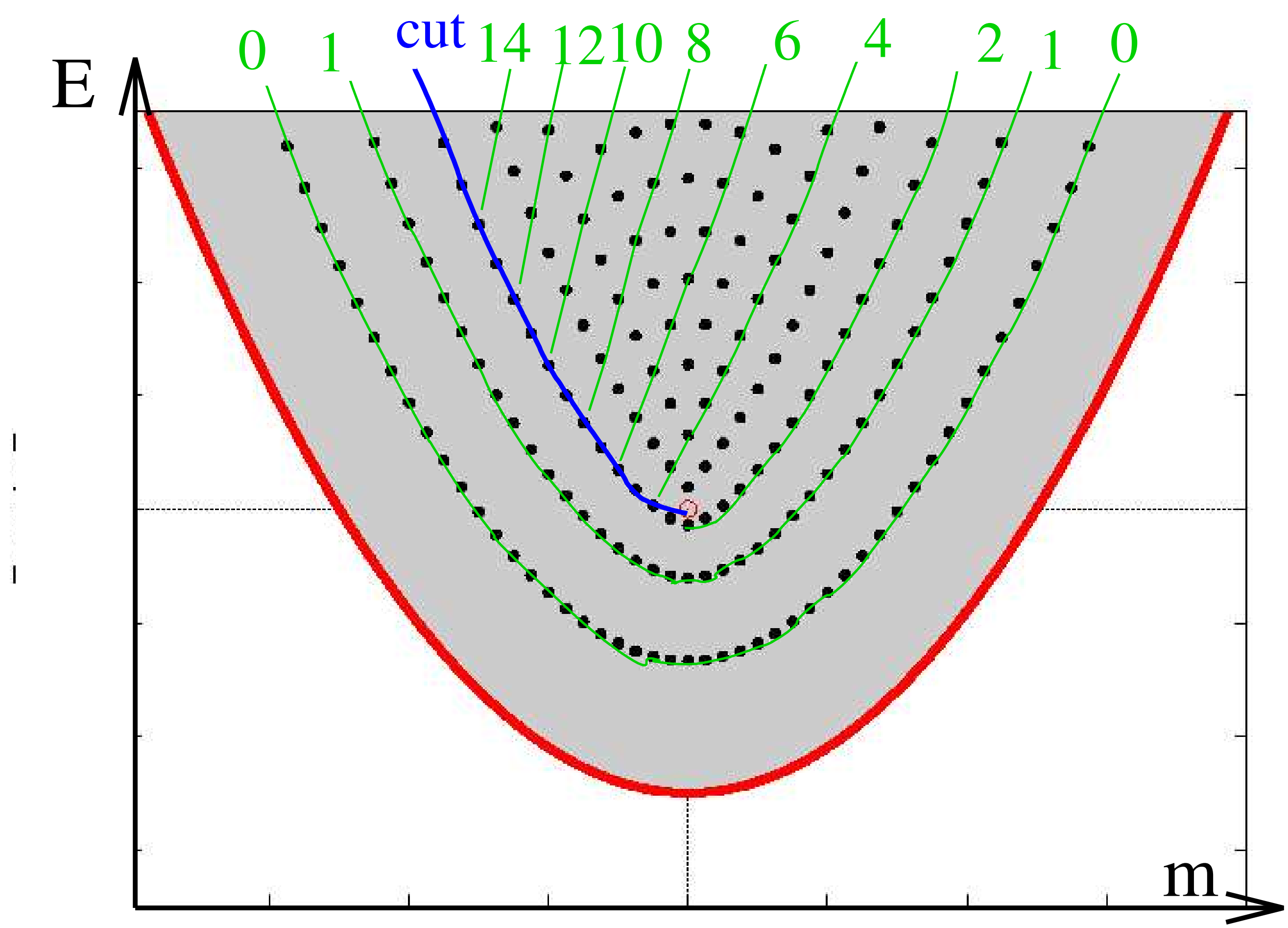

0
2
4
6
cut
6
4
2
0
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
Energie E
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
Composante z du moment cinétique m

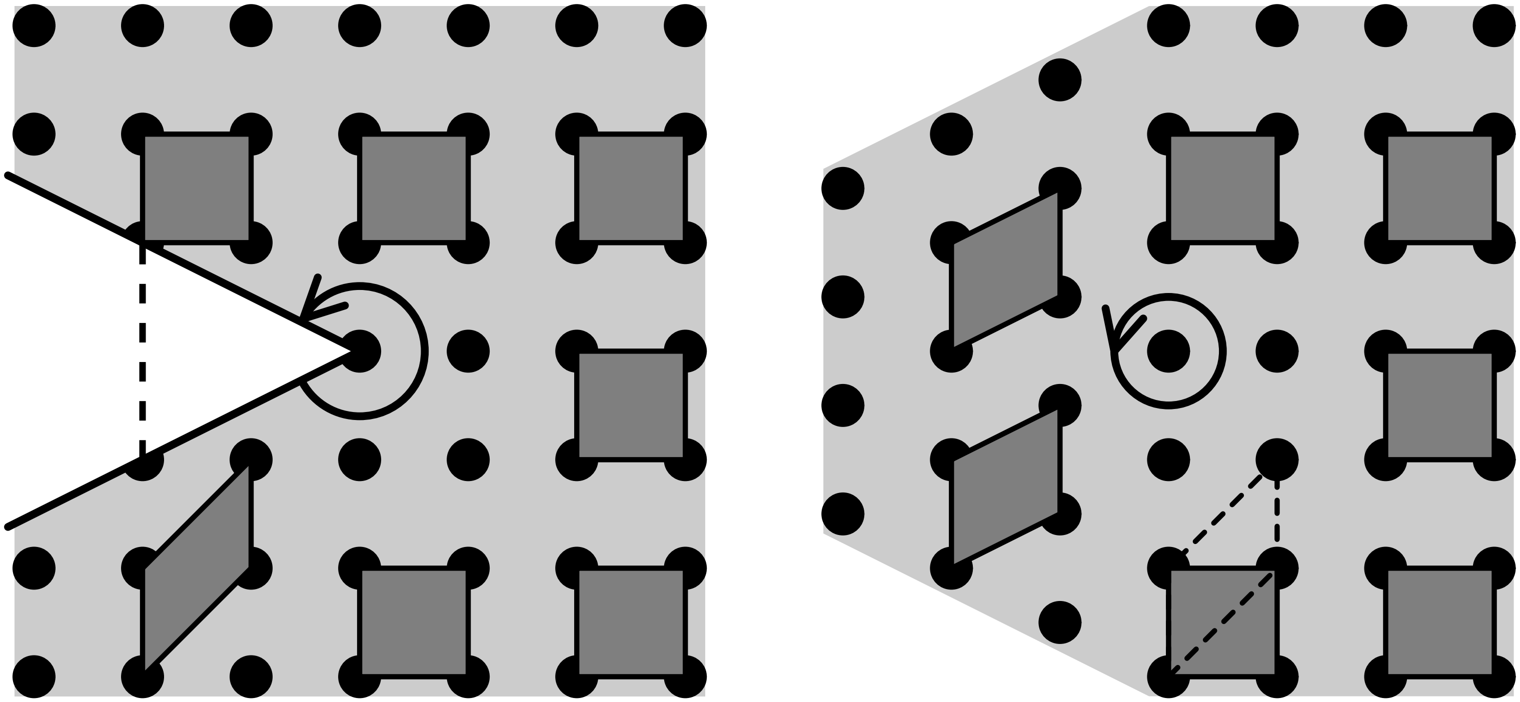
$1:(-1)$ resonant oscillator system with a cut along an *eigenray* represented in action variables.



“Eigenray“ is introduced by M.Symington [math.SG/0210033].

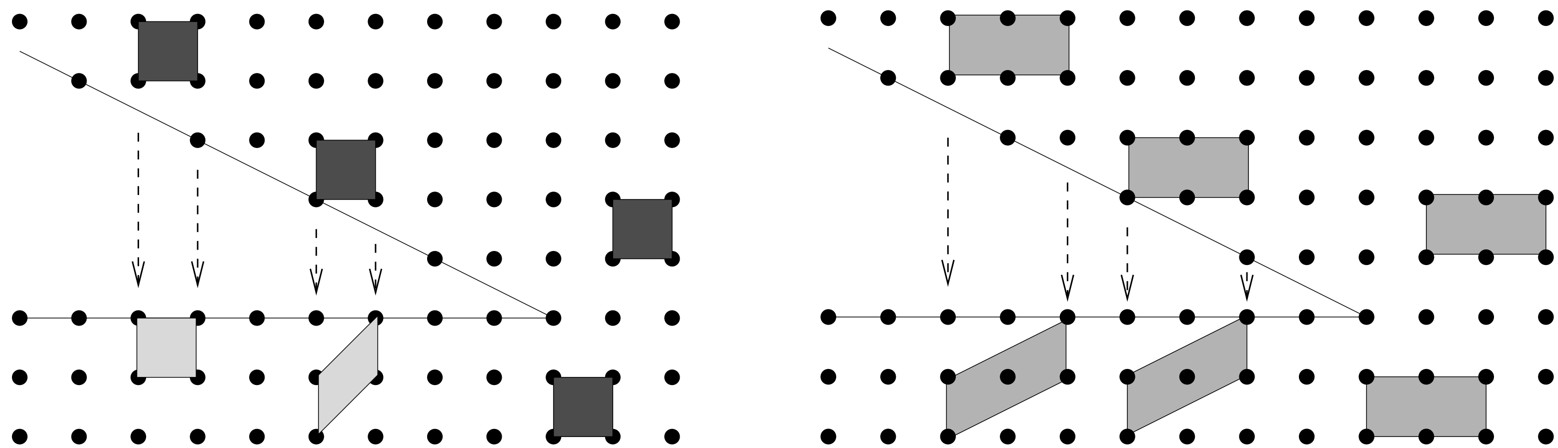
Line of “kink“ according to Child.


cut
0
1
14
12
10
8
6
4
2
1
0
E
m



Construction of the 1:(-1) lattice defect starting from the regular Z^2 lattice. Dark grey quadrangles show the evolution of an elementary lattice cell along a closed path around the defect point.

BZ, in “Topology in condensed matter“, Springer series in solid state sciences, vol. 150, 165-186 (2006)



Construction of 1 : 2 rational lattice defect.

Left: Elementary cell does not pass unambigously.

Right: Double cell passes.

N.Nekhoroshev, D. Sadovskii, BZ, Fractional Hamiltonian monodromy. Ann. Henri Poincaré, **7**, 1099-1211 (2006)

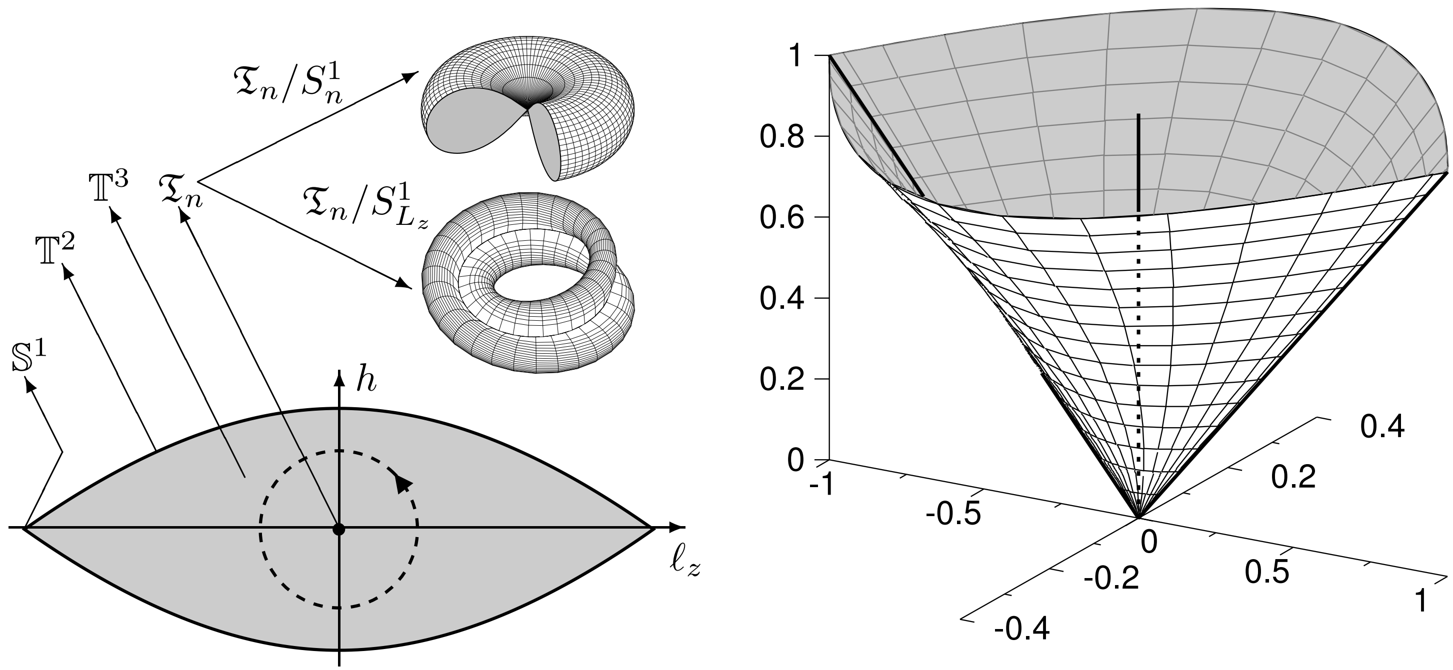
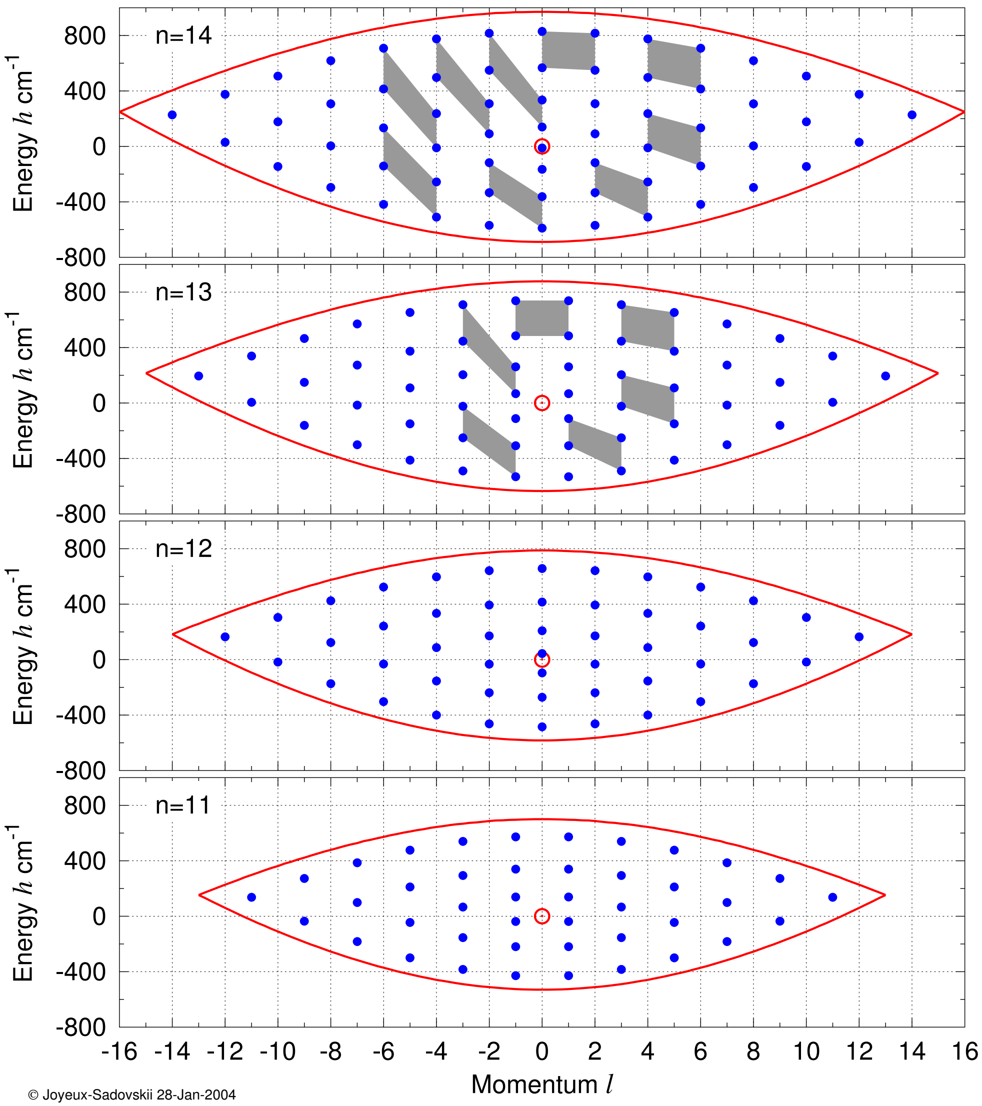
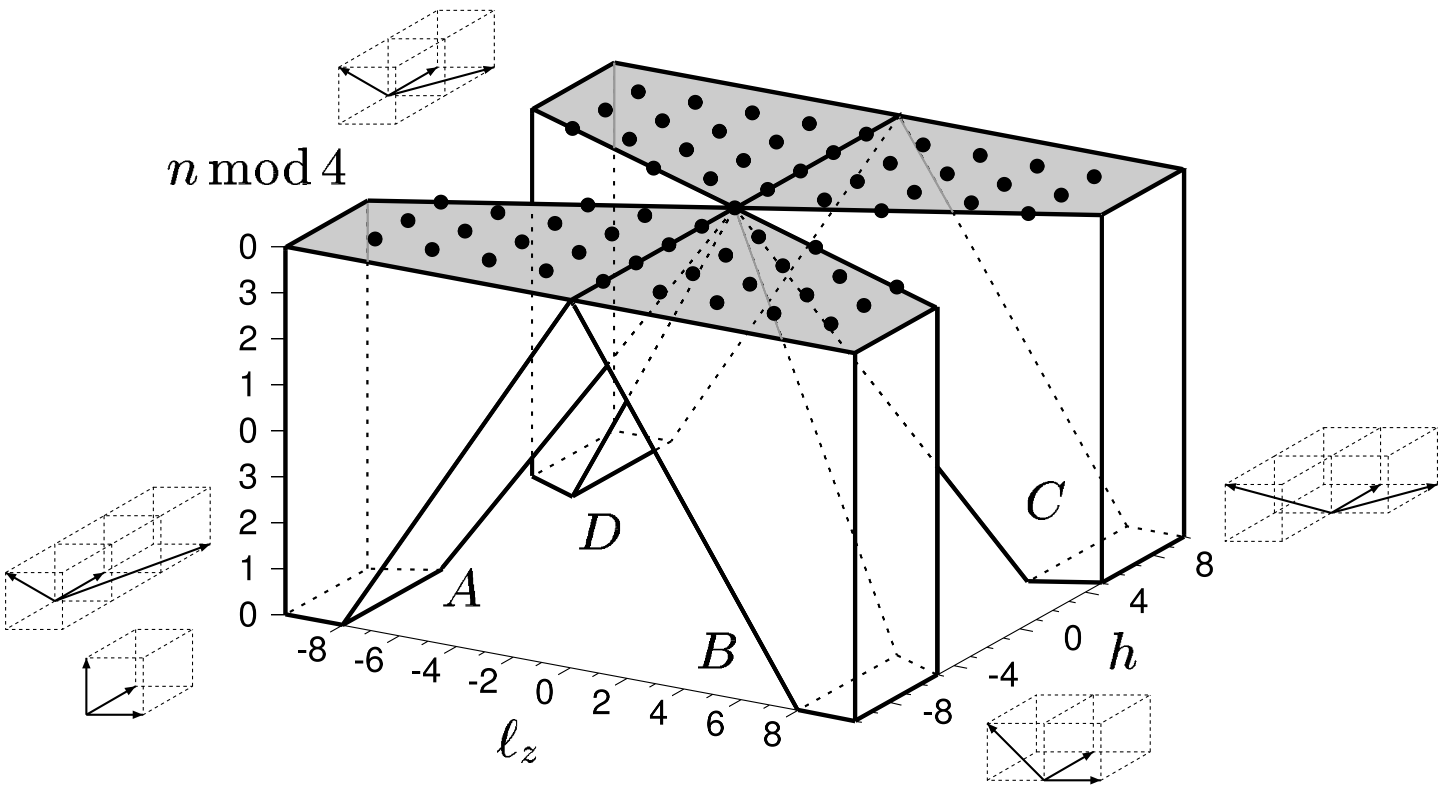


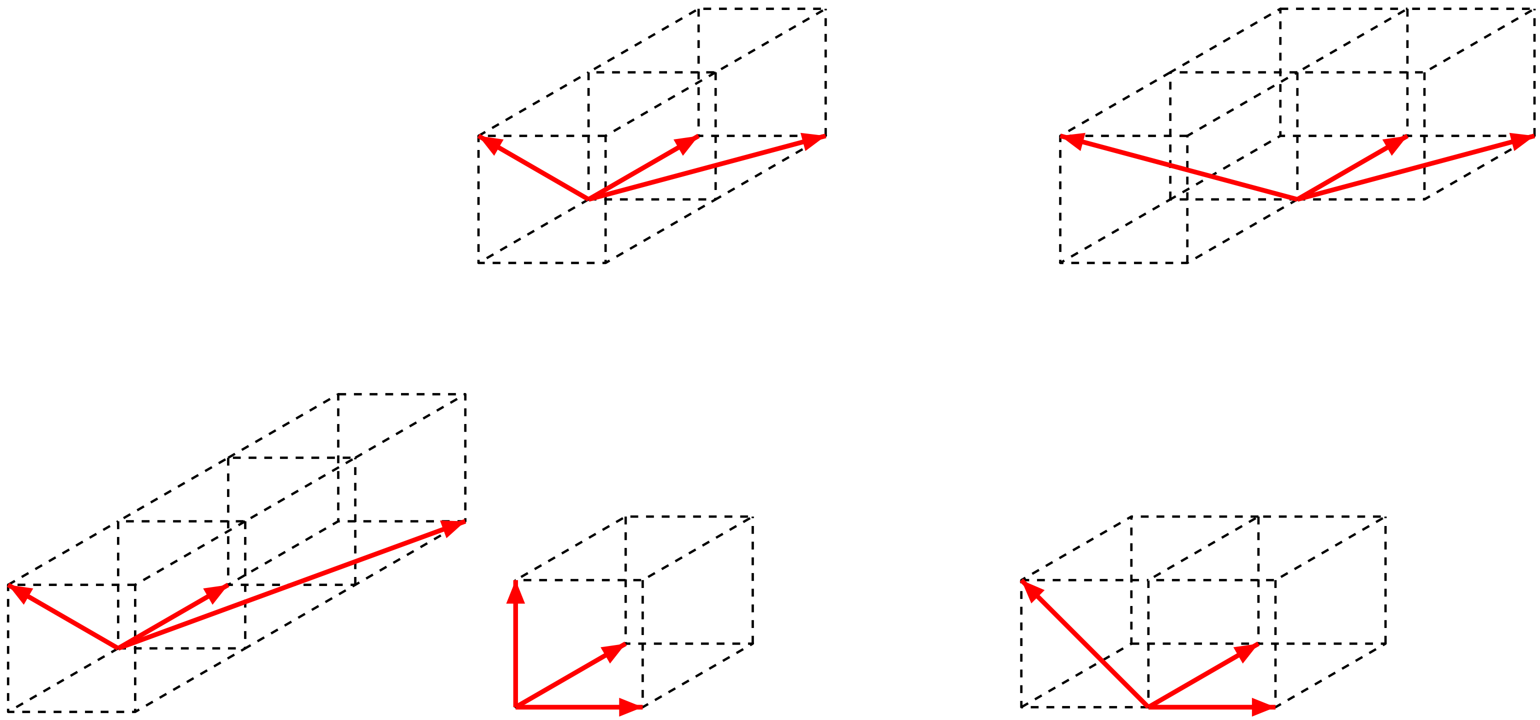
Image of the energy momentum map for the $1:1:2$ resonant oscillator system with axial symmetry (*and without detuning*). Full 3D-image, typical constant-n section and fibers.

R.H. Cushman *et al*, *Phys.Rev.Lett.* **93**, 024302, 2004


n=14
n=13
n=12
n=11
Energy h cm-1
800
400
0
-400
-800
-16 -14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16
Momentum l
© Joyeux-Sadovskii 28-Jan-2004



Quantum lattice with several elementary monodromy defects.



Matrix representation of monodromy for model with $1:1:2$ resonance

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

The monodromy matrix is defined up to similarity transformation $M \sim AMA^{-1}$ with $A \in SL(3,Z)$.

What defects (singularities) are physically interesting?

- for individual systems, for parametric families

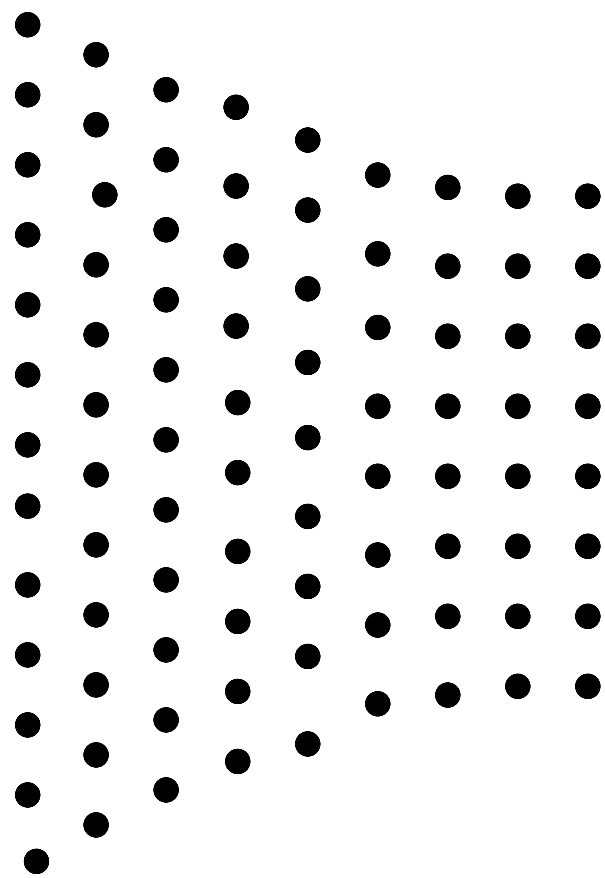
Does monodromy characterise defect?

- up to natural equivalence due to $SL(n, Z)$ conjugation

Could non-elementary defects be “generic“?

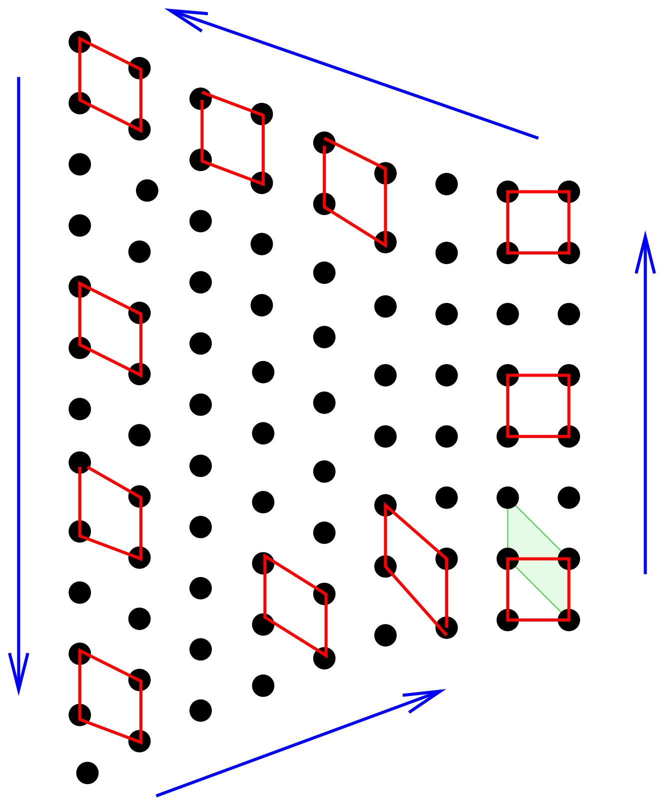
- equivariance due to higher “hidden“ symmetry

Is such pattern possible for Hamiltonian systems?



Violation of convexity

Different “sign“ of monodromy.



Compare with Duistermaat-Heckman theorem

see also: S.Vu Ngọc, Adv. Math. **208**, 909-934 (2007)

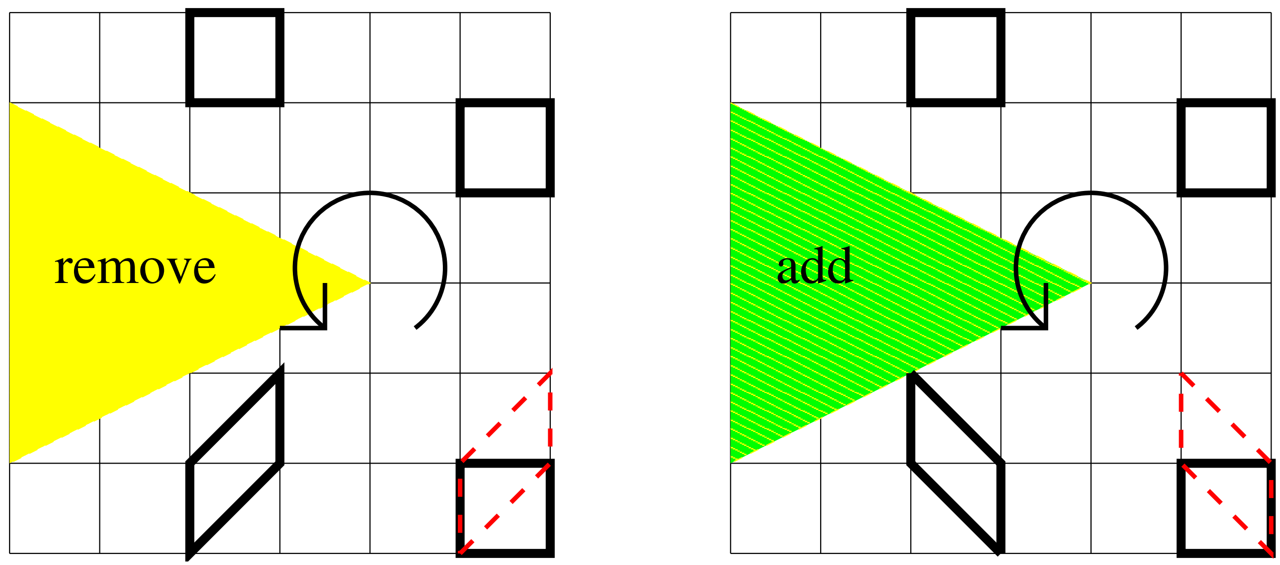
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

These two matrices belong to different classes of conjugated elements in $SL(2, Z)$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

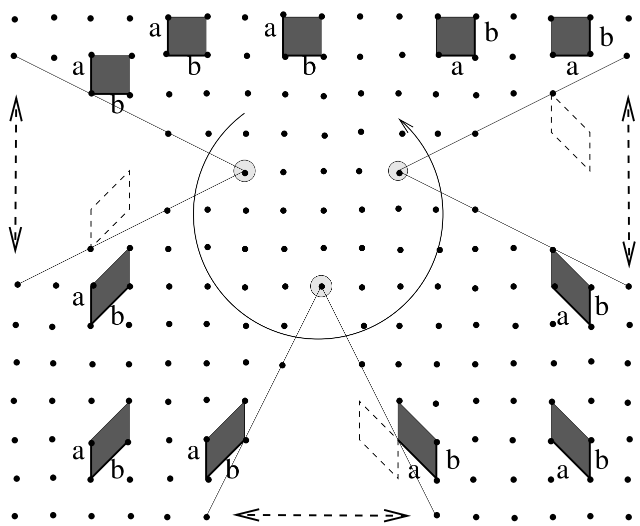
These two matrices belong to the same class of conjugated elements in $SL(3, Z)$.

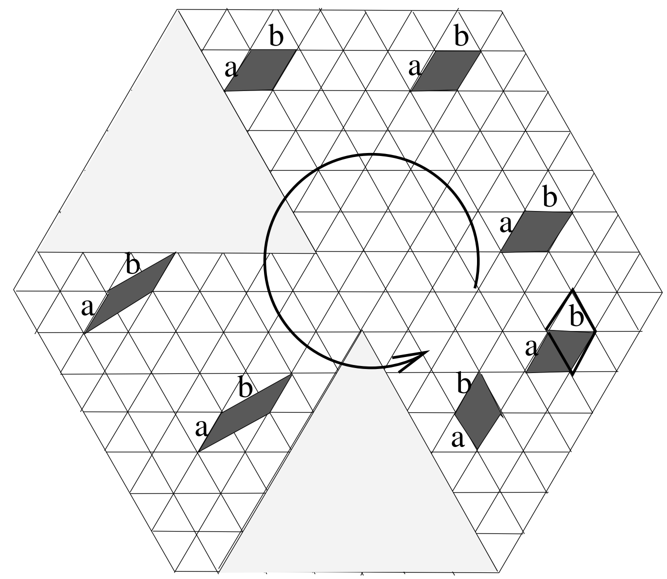
Comparison of elementary defects of different sign represented by *removing* or *adding* a wedge.



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \uparrow; \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \longrightarrow \qquad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

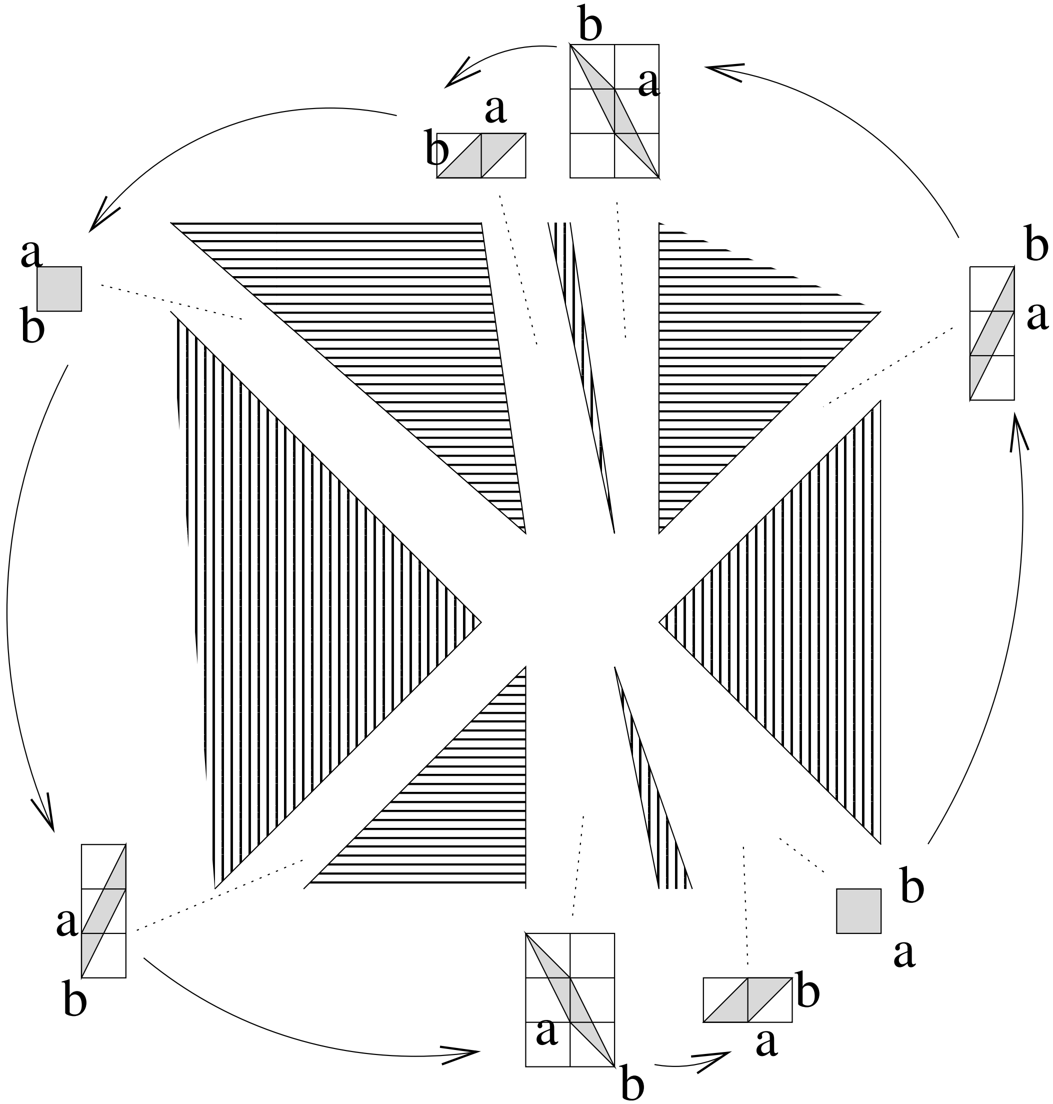
Multiple defects with the same sign (focus-focus singularities)





(Left) - Regular square lattice with three elementary monodromy defects.

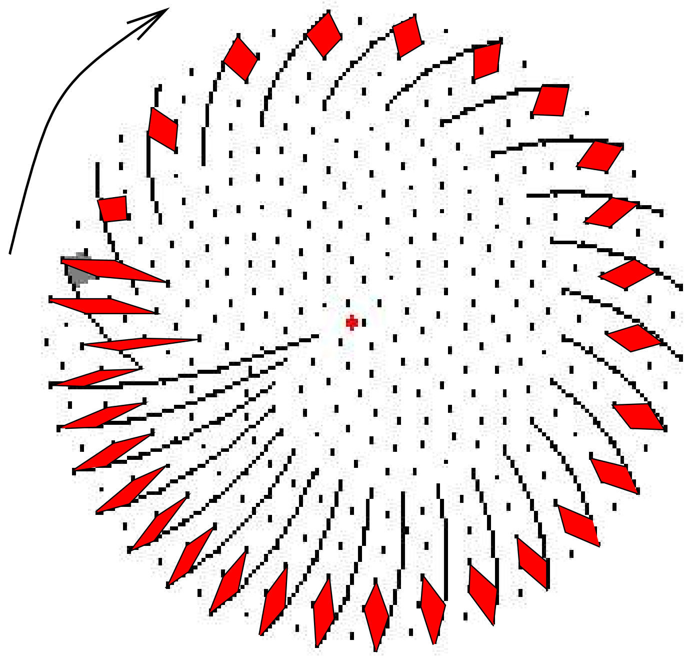
(Right) - Regular triangular lattice with two elementary monodromy defects.

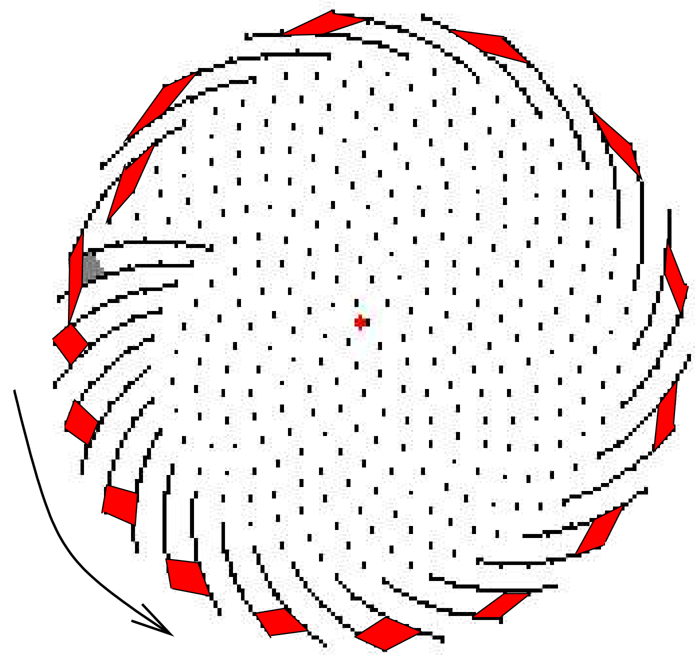


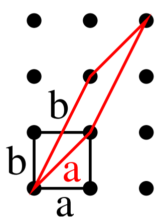
Cumulative effect of multiple (11) elementary defects.

“Sol-flower“ (Geodesic flow on *Sol*-Manifold)

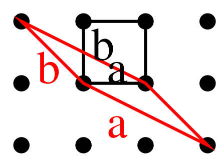
A.V.Bolsinov, H.R.Dullin, A.P.Veselov, *Spectra of Sol-Manifolds...*, Commun.Math.Phys. **264** 583-611 (2006); [figure 7 with my red color]







$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Sol-manifolds - T^2 torus bundles over a circle S^1 with hyperbolic gluing maps with positive eigenvalues.

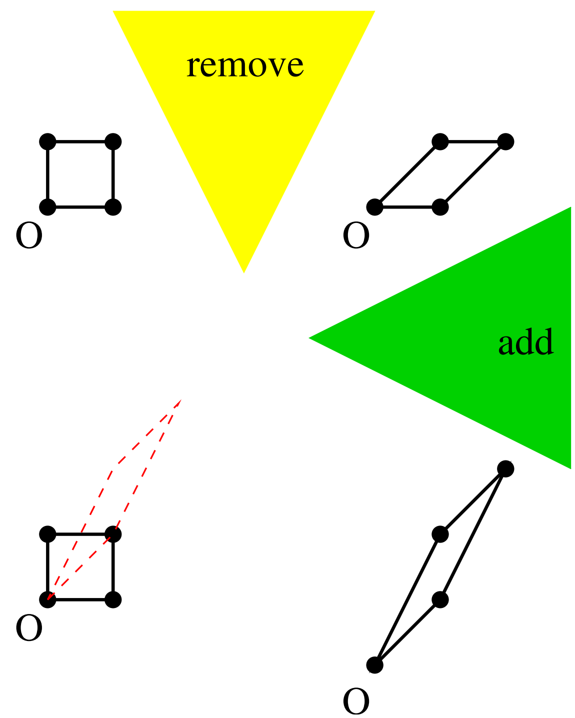
Let (x, y) be the standard periodic cordinates on T^2 defined modulo 1, $z \in (-\infty, +\infty)$ be a coordinate on R, and $\tilde{M}^3 = T^2 \times R$. Then the action generated on $\tilde{M}^3$ by transformation T_A

$$T_A : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longrightarrow \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ z + 1 \end{pmatrix}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2, Z)$ is an integer hyperbolic matrix, defines a hyperbolic automorphism of the 2-torus.

Sol-manifold M^3_A is defined as a quotient $\tilde{M}^3/Z$ by the action T_A.

Getting "cat map" monodromy $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ with one elementary positive and one elementary negative defects.



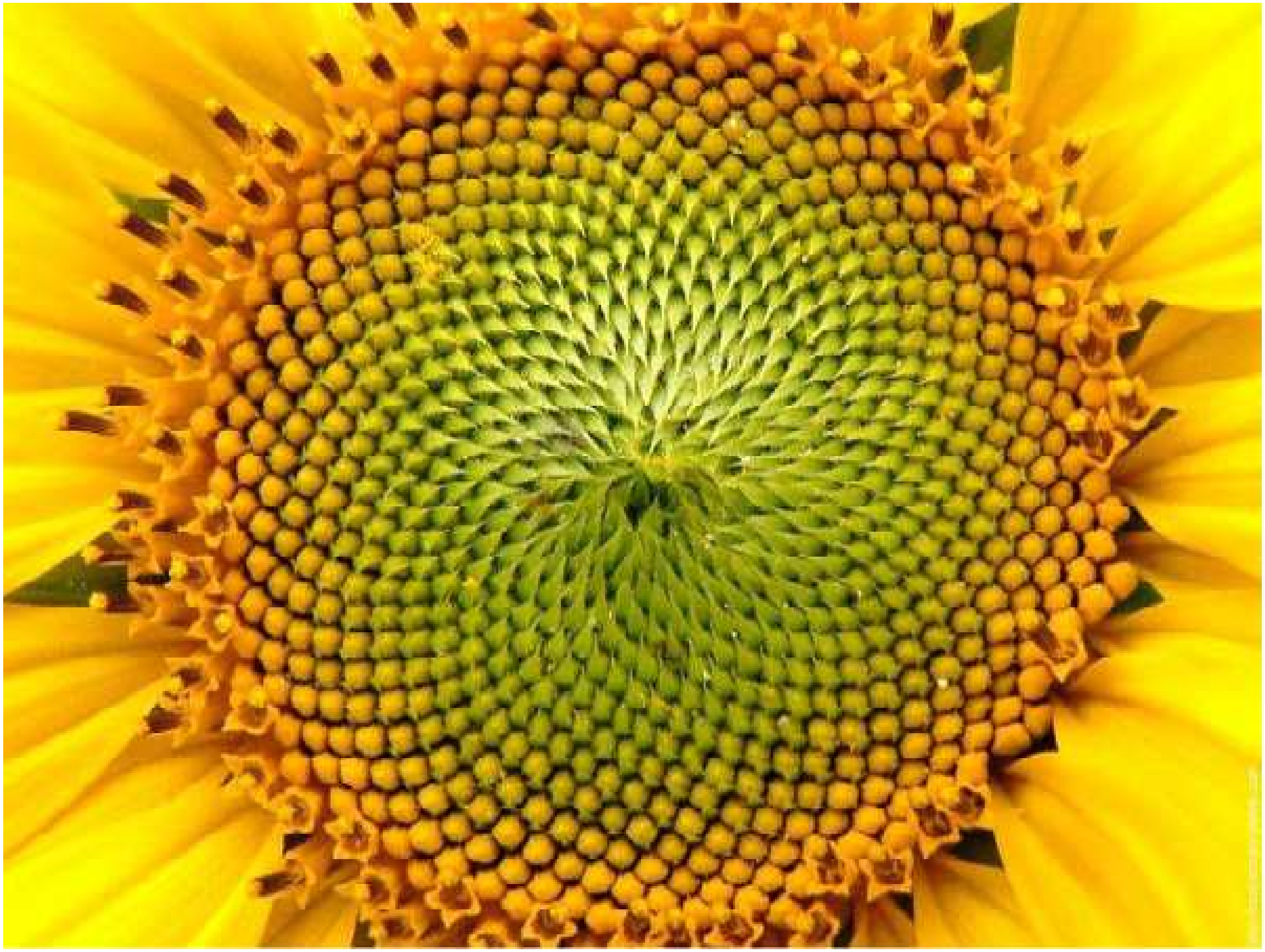
"No rotation" of elementary cell.

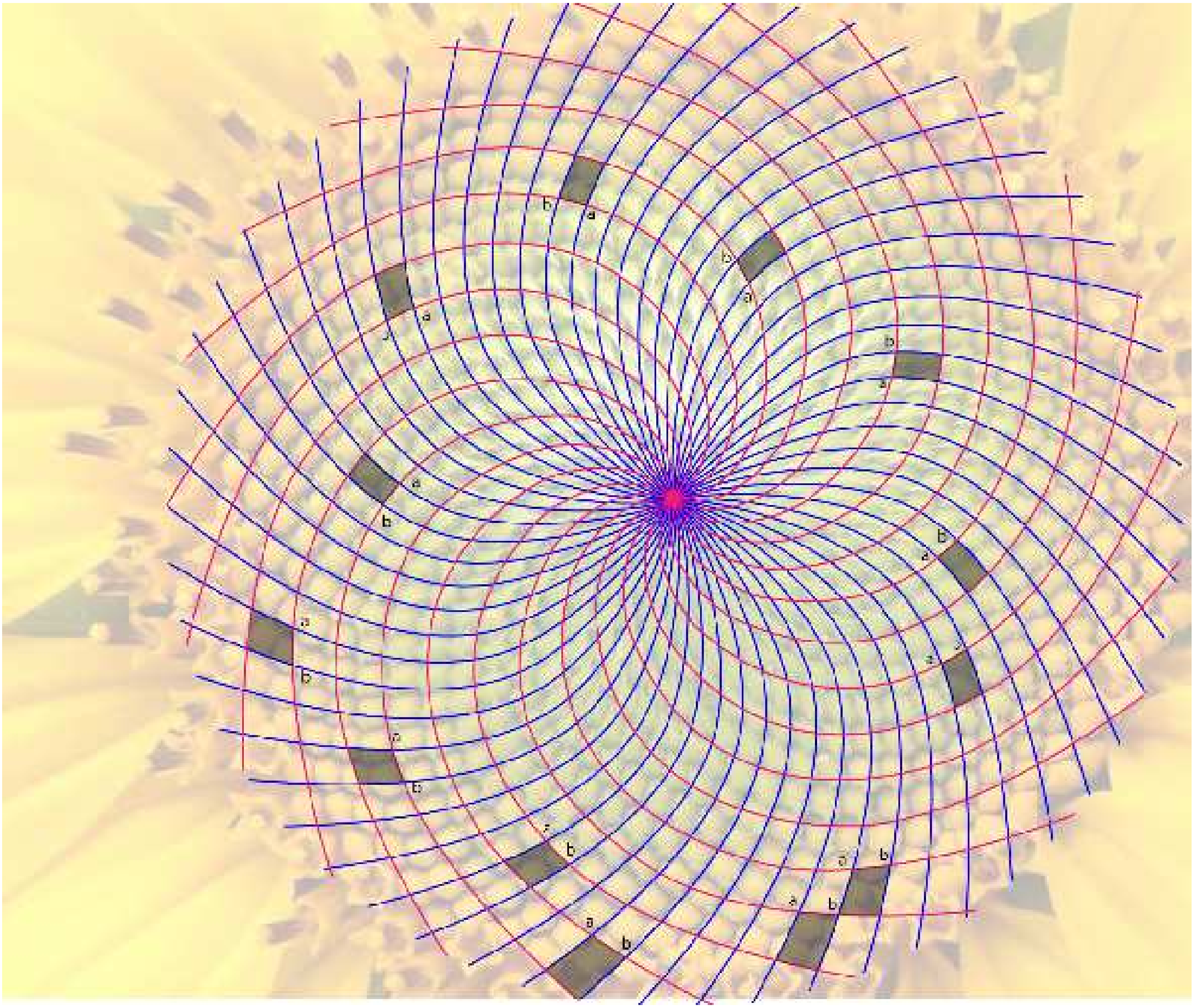
“Cat map” is related to Fibonacci numbers. Two consequitive Fibonacci numbers appear as components of a transformed basis vector after iterative application of the “cat map“.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad A^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix};$$

$$A^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}; \qquad \ldots$$




b
a
a
b
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b
a
b
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a
a
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b
a
b
a
b

CONCLUSIONS $\Leftrightarrow$ INTRODUCTION

Intuitive belief

Simple dynamical model with classical phase space being $K3$ space would be an important step towards description of new qualitative features in "complex systems".

Why $K3$?

Allows almost toric fibration with 24 isolated focus-focus singularities.

2 degree of freedom model dynamical system

In some way this is the simplest dynamical model on the chosen topologically non-trivial class of spaces.