# Generalised elliptic functions

**Chris Athorne and Matthew England**

School of Mathematics and Statistics
University of Glasgow

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# Outline



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# Strong connections with integrable PDEs



- KdV $\rightarrow$ translational symmetry reduction $\rightarrow$ Weierstrass $\wp$-function and a genus one curve.
- KdV with differential constraints $\rightarrow$ generalised $\wp$-functions and a genus $g$ hyperelliptic curve (e.g. next slide).
- Such solutions also arise via IST and $\theta$-functions.
- Interest for Integrable Systems to obtain equations satisfied by $\wp$-functions for general plane algebraic curves
- Wider interest due to applications in GR, Billiards, Statistical mechanics
- Interplay between function theory and geometry of curve

# Example: A genus two curve



Consider for $x, y \in \mathbb{C}$,

$$y^2 = \lambda_6 x^6 + 6\lambda_5 x^5 + 15\lambda_4 x^4 + 20\lambda_3 x^3 + 15\lambda_2 x^2 + 6\lambda_1 x + \lambda_0.$$

We have generalised $\wp$-functions - multiply periodic - denoted $\wp_{ij}(u_1, u_2)$ for $i, j = 1, 2$: Note $\wp_{ij} = \wp_{ji}$, $\partial_{u_i}\wp_{jk} = \partial_{u_j}\wp_{ik}$.

$$\begin{aligned}
-\tfrac{1}{3}\wp_{2222} + 2\wp_{22}^2 &= \lambda_4\wp_{22} - 2\lambda_5\wp_{12} + \lambda_6\wp_{11} + \lambda_2\lambda_6 - 4\lambda_3\lambda_5 + 3\lambda_4^2 \\
-\tfrac{1}{3}\wp_{1222} + 2\wp_{12}\wp_{22} &= \lambda_3\wp_{22} - 2\lambda_4\wp_{12} + \lambda_5\wp_{11} + \tfrac{1}{2}(\lambda_1\lambda_6 - 3\lambda_2\lambda_5 + 2\lambda_3\lambda_4) \\
-\tfrac{1}{3}\wp_{1122} + \tfrac{2}{3}\wp_{11}\wp_{22} + \tfrac{4}{3}\wp_{12}^2 &= \lambda_2\wp_{22} - 2\lambda_3\wp_{12} + \lambda_4\wp_{11} + \tfrac{1}{6}(\lambda_0\lambda_6 - 9\lambda_2\lambda_4 + 8\lambda_3^2) \\
-\tfrac{1}{3}\wp_{1112} + 2\wp_{11}\wp_{12} &= \lambda_1\wp_{22} - 2\lambda_2\wp_{12} + \lambda_3\wp_{11} + \tfrac{1}{2}(\lambda_0\lambda_5 - 3\lambda_1\lambda_4 + 2\lambda_2\lambda_3) \\
-\tfrac{1}{3}\wp_{1111} + 2\wp_{11}^2 &= \lambda_0\wp_{22} - 2\lambda_1\wp_{12} + \lambda_2\wp_{11} + \lambda_0\lambda_4 - 4\lambda_1\lambda_3 + 3\lambda_2^2
\end{aligned}$$

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# Example: A genus two curve continued



Consider again the first equation,

$$-\frac{1}{3}\wp_{2222}+2\wp_{22}^2=\lambda_4\wp_{22}-2\lambda_5\wp_{12}+\lambda_6\wp_{11}+\lambda_2\lambda_6-4\lambda_3\lambda_5+3\lambda_4^2$$

*Wlog* set $\lambda_6=0,\ \lambda_4=0,\ \lambda_5=\frac{1}{6}$, and make the change of variables $\wp_{22}=U(x,t),\ \partial_1=\partial_t,\ \partial_2=\partial_x$. Differentiate first identity *wrt* $x$ to find

$$U_t-U_{xxx}+12UU_x=0.$$

KdV, with four additional differential identities.

# The Riemann-Roch theorem



Let $X$ be a complex algebraic plane curve and $D$ a divisor on $X$:

$$D = n_1 P_1 + n_2 P_2 + \ldots + n_r P_r, \qquad P_i \in X,\ n_i \in \mathbb{Z}.$$

Riemann-Roch: $\dim H^0_D - \dim H^1_D = 1 - g + \deg D$

$$(\dim H^0_{D+P} - \dim H^0_D) + (\dim H^1_D - \dim H^1_{D+P}) = 1.$$

# The Riemann-Roch theorem: Example $g = 2$



Riemann-Roch: $\dim H^0_D - \dim H^1_D = 1 - g + \deg D$

$$(\dim H^0_{D+P} - \dim H^0_D) + (\dim H^1_D - \dim H^1_{D+P}) = 1.$$

E.g. genus two hyperelliptic - two Weierstraß *gaps* -

| | $\bullet$ | $\circ$ | $\bullet$ | $\circ$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet \cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $D$ | $0$ | $P$ | $2P$ | $3P$ | $4P$ | $5P$ | $6P$ | $7P$ | $8P$ | $9P$ | $10P$ |
| $\dim H^0_D$ | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $\dim H^1_D$ | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $1$ | | $x$ | | $x^2$ | $y$ | $x^3$ | $xy$ | $x^4$ | $x^2y$ | $x^5, y^2$ |
| | | $\downarrow$ | | $\downarrow$ | | | | | | | $\Downarrow$ |
| | | $\frac{dx}{f_y}$ | | $\frac{xdx}{f_y}$ | | | | | | | $f(x,y)$ |

The curve is a relation on $H^0_{10P}$:

$$f(x,y) = y^2 + [x^2]y + [x^5] = 0.$$

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# The Riemann-Roch theorem: Example $g = 3$



E.g. genus three non-hyperelliptic - three Weierstraß *gaps* -

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\bullet$ | $\circ$ | $\circ$ | $\bullet$ | $\bullet$ | $\circ$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ |
| $D$ | $0$ | $P$ | $2P$ | $3P$ | $4P$ | $5P$ | $6P$ | $7P$ | $8P$ | $9P$ | $10P$ | $11P$ | $12P$ |
| $\dim H^0_D$ | 1 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\dim H^1_D$ | 3 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| | $1$ | | | $x$ | $y$ | | $x^2$ | $xy$ | $y^2$ | $x^3$ | $x^2y$ | $xy^2$ | $x^4, y^3$ |
| | | $\downarrow$ $\frac{ydx}{f_y}$ | $\downarrow$ $\frac{dx}{f_y}$ | | | $\downarrow$ $\frac{xdx}{f_y}$ | | | | | | | $\Downarrow$ $f(x,y)$ |

The curve is a relation on $H^0_{12P}$:

$$f(x,y) = y^3 + [x]y^2 + [x^2]y + [x^4] = 0.$$

In each case $x$ and $y$ are coordinates on $X$, i.e. maps $X \to \mathbb{P}^1$.
Interested later in coordinate transformations
$x \mapsto \tilde{x} = \psi \circ x$ for $\psi : \mathbb{P}^1 \to \mathbb{P}^1$.

# Definitions for generic genus: Abel Map



Let $H_0^0 =< \omega_1, \ldots, \omega_g >$ be basis of holomorphic 1-forms on X.
Define the Abel map:

$$\phi : \; Symm^g X \to \mathbb{C}^g \to \mathbb{C}^g/\Pi = Jac(X)$$

$$P_1 + .. + P_g \mapsto \sum_{i=1}^{g} \int_{P_0}^{P_i} \omega \, (mod \, \Pi)$$

- $D$ is principle $\Leftrightarrow$ degree $D$=0 and $\phi(D) = 0$
- $\phi$ is surjective
- Inversion problem: " find $D$ given a point in Jac(X)"

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# Definitions for generic genus: Theta Functions



The (simplified) $\theta$ function is entire on $\mathbb{C}^g \ni u$

$$\theta(u, \tau) = \sum_{N \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} N^t \tau N + N^t \tau u \right)$$

$\tau$ a complex symmetric $g \times g$ matrix related to periods.

- $\theta$ has modular properties implying that its second logarithmic derivatives *wrt* $u$ are well-defined on $Jac(X)$
- zeros of $\theta \circ \phi$ allow us to reconstruct $P_1, \ldots, P_g$
- $\sigma$ function is a modified $\theta$ function used to define the Weierstrass $\wp$-functions: $\wp_{ij} = -\partial_i \partial_j \ln \sigma$.
- $\Theta$-divisor is locus of zeros of $\theta$. The $\sigma$ function has zero of first order at the non-singular points of $\Theta$-divisor $\Rightarrow$ order two poles in the $\wp_{ij}$.

# Outline



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# Abelian functions associated with algebraic curves 

**Definition**

A cyclic $(n,s)$-curve is an algebraic curve with equation

$$y^n = [x^s] \equiv x^s + \lambda_{s-1}x^{s-1} + \ldots + \lambda_1 x + \lambda_0, \qquad \lambda_j \text{ constants},$$

where we choose $(n,s)$ coprime with $n < s$. These curves have a unique branch point at infinity and genus $g = \frac{1}{2}(n-1)(s-1)$.

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# Abelian functions associated with algebraic curves 1/11

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An Abelian function associated to a curve is a meromorphic function of $\boldsymbol{u} \in \mathbb{C}^g$ that is periodic on the lattice generated by the period matrices of the curve, $\omega_1$ and $\omega_2$. I.e. $\mathfrak{M}(\boldsymbol{u})$ satisfying

$$\mathfrak{M}(\boldsymbol{u} + \omega_1\boldsymbol{n} + \omega_2\boldsymbol{m}) = \mathfrak{M}(\boldsymbol{u}),$$

for all integer vectors $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g$.

# Kleinian $\wp$-functions



## Kleinian $\wp$-functions

Define the Kleinian $\wp$-functions as the second log derivatives of the multivariate $\sigma$-function, $\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, ..., u_g)$ associated to some algebraic curve:

$$\wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(\mathbf{u}), \quad i \leq j \in \{1, 2, ..., g\}$$

We can extend this notation to higher order derivatives:

$$\wp_{ijk} = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} \ln \sigma(\mathbf{u}), \qquad i \leq j \leq k \in \{1, 2, ..., g\}$$

etc. They are all Abelian functions.

# The Sato weights



For every curve we define a set of Sato weights that render the associated Abelian function theory homogeneous.

Example: Consider the cyclic (4,5)-curve, which has $g = 6$.

$$y^4 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

In this case the Sato weights are

| $x$ | $y$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $\lambda_4$ | $\lambda_3$ | $\lambda_2$ | $\lambda_1$ | $\lambda_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -4 | -5 | 11 | 7 | 6 | 3 | 2 | 1 | -4 | -8 | -12 | -16 | -20 |

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**Example:** $\mathrm{wt}(\lambda_3 x^3) = -8 + 3(-4) = -20$

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# The sigma-function expansion



- We want to construct a power series expansion for $\sigma(\boldsymbol{u})$.
- The canonical limit of the $(n,s)$ $\sigma$-function is equal to the Schur-Weierstrass polynomial generated by $(n,s)$.

$$\sigma(\mathbf{u};\boldsymbol{\lambda})\Big|_{\boldsymbol{\lambda}=\mathbf{0}} = \sigma(\mathbf{u};0) = SW_{n,s}$$

Buchstaber, Enolski, Leykin
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- We hence know the start of the expansion and the weight of $\sigma(\boldsymbol{u})$. The expansions will be multivariate expansion in $\boldsymbol{u} = (u_1, u_2, \dots, u_g)$ but could also depend on curve coefficients, $\{\lambda_0, \lambda_1, \dots \lambda_s\}$.

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# The sigma-function expansion II



In the (4,5)-case the $\sigma$-expansion has weight +15, and contains $u_i$ and $\lambda_j$. So we write the expansion as

$$\sigma(\boldsymbol{u}) = C_{15} + C_{19} + C_{23} + \ldots + C_{15+4i} + \ldots$$

where each $C_k$ has weight $k$ in the $u_i$ and $(15 - k)$ in the $\lambda_j$. We have $C_{15} \equiv SW_{4,5}$.

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where each $C_k$ has weight $k$ in the $u_i$ and $(15 - k)$ in the $\lambda_j$. We have $C_{15} \equiv SW_{4,5}$. Find the other $C_k$ in turn by:

1. Identifying the possible terms — those with correct weight.
2. Form the sigma function with unidentified coefficients.
3. Determine coefficients by satisfying known properties.

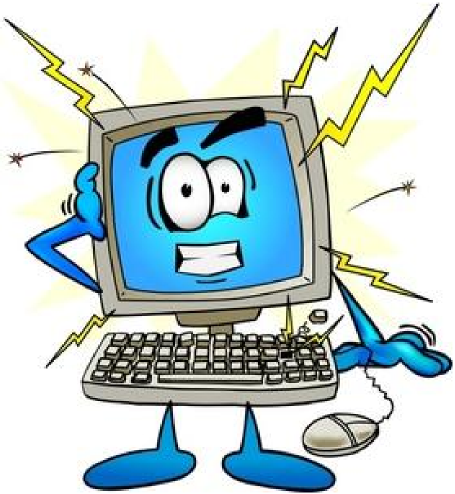
# Deriving properties of the functions



The $\sigma$-expansion can be used as an experimental tool to derive a variety of properties for both the $\sigma$- and $\wp$-functions such as:

- Differential equations satisfied by the $\wp$-functions.
- Addition formulae for the sigma-functions.
- Identification of basis for space of functions.

Significant simplifications in computations can be achieved through careful coding.



# Computations can be large!

| Polynomial | # Terms |
|---|---|
| $C_{19}$ | 50 |
| $C_{27}$ | 386 |
| $C_{35}$ | 2193 |
| $C_{43}$ | 8463 |
| $C_{51}$ | 28359 |
| $C_{59}$* | 81832 |

∗ # possible terms = 120964

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# Bases of Abelian functions



Let $\Gamma(m)$ = the space of Abelian functions with poles of order at most $m$. We seek bases for such spaces.

1. The Riemann-Roch theorem for Abelian varieties gives $\dim(\Gamma(m)) = g^m$.
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4. We include the function $\Delta = \wp_{11}\wp_{22} - \wp_{12}^2$ in $\Gamma(3)$. When $g = 2$, subsequent bases are identified using steps 1-3.

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4. We include the function $\Delta = \wp_{11}\wp_{22} - \wp_{12}^2$ in $\Gamma(3)$. When $g = 2$, subsequent bases are identified using steps 1-3.
5. For $g > 2$ we need new candidate functions...

# Deriving bases I



We have two approaches to find candidate functions for $\Gamma(m)$. In either case linear independence must be checked using the $\sigma$-expansion.

1. Matching poles in algebraic combinations of $\wp$-functions, analogous to $\Delta \in \Gamma(3)$.

For example,

$$\mathcal{B}_{ijklm} = \wp_{ij}\wp_{klm} + \frac{1}{3}\big(\wp_{jk}\wp_{ilm} + \wp_{jl}\wp_{ikm} + \wp_{jm}\wp_{ikl} - 2\wp_{kl}\wp_{ijm} - 2\wp_{km}\wp_{ijl} - 2\wp_{lm}\wp_{ijk}\big)$$

always belongs to $\Gamma(3)$.

# Deriving bases II



2. Applying differential operators on $\sigma$-functions to leave Abelian functions.

## The $Q$-functions

Define Hirota's bilinear operator as $\mathcal{D}_i = \partial/\partial u_i - \partial/\partial v_i$. Then define the *m*-index *Q*-functions (for $m$ even).

$$Q_{i_1,i_2,\ldots,i_m}(\boldsymbol{u}) = \frac{(-1)}{2\sigma(\boldsymbol{u})^2}\mathcal{D}_{i_1}\mathcal{D}_{i_2}\ldots\mathcal{D}_{i_m}\sigma(\boldsymbol{u})\sigma(\boldsymbol{v}) \; \Big|_{\boldsymbol{v}=\boldsymbol{u}}$$

$$i_1 \leq \ldots \leq i_n \in \{1,\ldots,g\}.$$

These are all Abelian functions with poles of order two and so can be used to construct bases of $\Gamma(2)$.

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# Two-term addition formulae



**Theorem**

*Every $(n, s)$-curve has an associated addition formula*

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2} = \sum_i c_i A_i(\boldsymbol{u}) B_i(\boldsymbol{v})$$

*where $A_i, B_i \in \Gamma(2)$ and the $c_i$ are constants.*

- Determine exact formula using bases and $\sigma$-expansion.

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*where $A_i, B_i \in \Gamma(2)$ and the $c_i$ are constants.*

- Determine exact formula using bases and $\sigma$-expansion.
- Use available simplifications; symmetry, parity, weights.
- Generalise classic Weierstrass formula,

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(v) - \wp(u).$$

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# Automorphism addition formulae



There are a second class of addition formulae for cyclic curves,

$$y^n = x^s + \lambda_{s-1}x^{s-1} + \ldots + \lambda_1 x + \lambda_0$$

associated with their extra family of automorphisms:

$$[\zeta^j] : (x, y) \to (x, \zeta^j y), \qquad \text{where } \zeta^n = 1.$$

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The following function will be Abelian in $\boldsymbol{u}^{[i]}, i = 1 \dots n$:

$$\prod_{j=1}^{n} \frac{\sigma\left(\sum_{i=1}^{n} [\zeta^{i+j}] \boldsymbol{u}^{[i]}\right)}{\sigma((\boldsymbol{u}^{[j]})^n)}$$

So we can write is as sum of products of $n$ functions from $\Gamma(n)$ in the variables, $\boldsymbol{u}^{[i]}, i = 1 \dots n$ respectively.

# Simplified and reduced addition formulae

Such formulae can be difficult to compute. Simplified versions may be found, setting one or more of the variables $\boldsymbol{u}^{[i]}$ to zero.

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We can also consider reduced curves which have further automorphisms and hence extra addition formulae.

**Example:** The restricted (3,4)-curve, $y^3 = x^4 + \lambda_0$ has automorphisms

$$[i^j] : (x, y) \mapsto ((i)^j x, y), \qquad \text{where i is the complex variable.}$$

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$$[i^j] : (x, y) \mapsto ((i)^j x, y), \qquad \text{where i is the complex variable.}$$

The functions associated to this curve satisfy

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}+[i]\boldsymbol{v})\sigma(\boldsymbol{u}+[i^2]\boldsymbol{v})\sigma(\boldsymbol{u}+[i^3]\boldsymbol{v})}{\sigma(\boldsymbol{u})^4\sigma(\boldsymbol{v})^4} = f(\boldsymbol{u},\boldsymbol{v}) - f(\boldsymbol{v},\boldsymbol{u})$$

where $f$ is composed of functions in $\Gamma(4)$.

# Outline

1. Starting points
   - Motivations
   - Riemann-Roch Theorem
   - Definition of higher genus functions
2. Working with sigma-expansions
   - Functions, weights and expansions
   - Bases
   - Deriving addition formulae
3. The equivariant approach
   - Equivariant curves and functions
   - New results
4. Future prospects

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# The main idea



Equivariant theory attempts to:

- Shortcut some of the calculational intricacies of the $\sigma$-expansion.
- Keep a firm grip of the underlying geometry.

Choose simple $PSL(2,\mathbb{C})$ coordinate transformations on the curve coordinates

$$x \to \tilde{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad y \to \tilde{y} = \frac{y}{(\gamma x + \delta)^p}$$

Then $H_0^1$ becomes a $g$-dimensional $SL(2,\mathbb{C})$ module which may be reducible.

# Decompositions



- Hyperelliptic (2,2g+1) cases (genus $g$), $H_0^1$ is a **g** and the $\wp_{ij}$ decompose as

$$\mathbf{g} \odot \mathbf{g} = (\mathbf{2g-1}) \oplus (\mathbf{2g-5}) \oplus \ldots$$

- Trigonal (3,3p+1) cases (genus $3p$), $H_0^1$ is a $\mathbf{p} \oplus \mathbf{2p}$ and the $\wp_{ij}$ decompose as

$$\mathbf{p} \odot \mathbf{p} = (\mathbf{2p-1}) \oplus (\mathbf{2p-5}) \oplus \ldots$$

$$\mathbf{p} \otimes \mathbf{2p} = (\mathbf{3p-1}) \oplus (\mathbf{3p-3}) \oplus (\mathbf{3p-5}) \ldots$$

$$\mathbf{2p} \odot \mathbf{2p} = (\mathbf{4p-1}) \oplus (\mathbf{4p-5}) \oplus \ldots$$

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# Identities



The $\wp$-function identities fall into (finite dimensional) irreducible modules classified by highest weight elements. Strategy:

- Singularity expansion $\rightarrow$ a highest *weight* identity.
- Form module of identities.
- Further singularity expansion modulo this module (just linear algebra) $\rightarrow$ further identities.
- Tools - use of $\mathfrak{sl}(2, \mathbb{C})$ to locate highest weight elements and *Casimir* to identify dimensions.

# Explicit Results I



Analogue for the Weierstraß cubic in the hyperelliptic genus three case.

$$P = \begin{bmatrix} 0 & 0 & \wp_{11} & \wp_{12} & \wp_{13} \\ 0 & -2\wp_{11} & -\wp_{12} & \wp_{22} - 2\wp_{13} & \wp_{23} \\ \wp_{11} & -\wp_{12} & 2\wp_{13} - 2\wp_{22} & -\wp_{23} & \wp_{33} \\ \wp_{12} & \wp_{22} - 2\wp_{13} & -\wp_{23} & -2\wp_{33} & 0 \\ \wp_{13} & \wp_{23} & \wp_{33} & 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} \lambda_0 & 4\lambda_1 & 6\lambda_2 & 4\lambda_3 & \lambda_4 \\ 4\lambda_1 & 16\lambda_2 & 24\lambda_3 & 16\lambda_4 & 4\lambda_5 \\ 6\lambda_2 & 24\lambda_3 & 36\lambda_4 & 24\lambda_5 & 6\lambda_6 \\ 4\lambda_3 & 16\lambda_4 & 24\lambda_5 & 16\lambda_6 & 4\lambda_7 \\ \lambda_4 & 4\lambda_5 & 6\lambda_6 & 4\lambda_7 & \lambda_8 \end{bmatrix}$$

# Explicit Results II



$$A = \begin{bmatrix} 0 & -\wp_{333} & \wp_{233} & -\wp_{223} + \wp_{133} & \wp_{222} - 2\wp_{123} \\ \wp_{333} & 0 & -\wp_{133} & \wp_{123} & -\wp_{122} + \wp_{113} \\ -\wp_{233} & \wp_{133} & 0 & -\wp_{113} & \wp_{112} \\ \wp_{223} - \wp_{133} & -\wp_{123} & \wp_{113} & 0 & -\wp_{111} \\ -\wp_{222} + 2\wp_{123} & \wp_{122} - \wp_{113} & -\wp_{112} & \wp_{111} & 0 \end{bmatrix}$$

# Explicit Results III



Then

$$(\mathbf{l}^T A\mathbf{k})(\mathbf{l}'^T A\mathbf{k}') + \frac{1}{4}\det\begin{bmatrix} H-2P & \mathbf{l}^T & \mathbf{k}^T \\ \mathbf{l}' & 0 & 0 \\ \mathbf{k}' & 0 & 0 \end{bmatrix} = 0$$

where the $\mathbf{l}$ etc. are 5-vectors of arbitrary parameters.
For comparison, consider the genus one case (n.b. $\wp_{111} \equiv \wp'$)

$$\wp_{111}^2 + \frac{1}{4}\begin{vmatrix} \lambda_0 & 2\lambda_1 & \lambda_2 - 2\wp_{11} \\ 2\lambda_1 & 4\lambda_2 + 4\wp_{11} & 2\lambda_3 \\ \lambda_2 - 2\wp_{11} & 2\lambda_3 & \lambda_4 \end{vmatrix} = 0$$

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# Notes



- $A$ is of rank two $\Rightarrow$ “Plücker" relations on the $\wp_{ijk}$. Correspondence between the entries of $A$ and Plücker coordinates on G(2,5) which suggests natural generalizations of this case to higher (hyperelliptic) genus.
- Non-hyperelliptic cases still under construction.
- The known formulae for other cases given earlier are equivariant at leading order in the $\wp_{ij\ldots k}$ but not being derived from fully equivariant models of $X$ the lower order details need finessing.

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# Outline



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# Future prospects



Aim to fuse the approaches, starting with bases. E.g. $g = 2$:

| **Space** | **dim** | **basis** |
|---|---|---|
| $\Gamma(0)$ | 1 | $\{1\}$ |
| $\Gamma(1)$ | 1 | $\{1\}$ |
| $\Gamma(2)$ | 4 | $\{1, \wp_{11}, \wp_{12}, \wp_{22}\}$ |
| $\Gamma(3)$ | 9 | $\{1, \wp_{11}, \wp_{12}, \wp_{22},$ $\wp_{111}, \wp_{112}, \wp_{122}, \wp_{222}, \Delta = \wp_{11}\wp_{22} - \wp_{12}^2\}$ |

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| $\Gamma(4)$ | 16 | $\{1, \ldots, \Delta, \wp_{1111}, \ldots, \wp_{2222}, \partial_1\Delta, \partial_2\Delta\}$ |

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| $\Gamma(4)$ | 16 | $\{1, \ldots, \Delta, \wp_{1111}, \ldots, \wp_{2222}, \partial_1\Delta, \partial_2\Delta\}$ |
| $\vdots$ | $\vdots$ | |
| $\Gamma(m)$ | $m^2$ | $\{$basis for $\Gamma(m-1), \{\wp_{i_1 \ldots i_m}\}, \{\partial_{i_1} \ldots \partial_{i_{m-2}}\Delta\}$ |
| | $m^2$ | $= \quad (m-1)^2 + (m+1) + (m-2)$ |

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# Future prospects



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| Space | dim | basis |
|---|---|---|
| $\Gamma(0)$ | 1 | $\{1\}$ |
| $\Gamma(1)$ | 1 | $\{1\}$    $\underline{3}$    $\underline{3} \odot \underline{3} = \underline{5} \oplus \underline{1}$ |
| $\Gamma(2)$ | 4 | $\{1, \wp_{11}, \wp_{12}, \wp_{22}\}$ |
| $\Gamma(3)$ | 9 | $\{1, \wp_{11}, \wp_{12}, \wp_{22}, \wp_{111}, \wp_{112}, \wp_{122}, \wp_{222}, \Delta = \wp_{11}\wp_{22} - \wp_{12}^2\}$ |
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| $\Gamma(0)$ | 1 | $\{1\}$ |
| $\Gamma(1)$ | 1 | $\{1\}$ $\underline{3}$ $\quad \partial \odot \underline{n} = \underline{n+1} \oplus \underline{n-1}$ |
| $\Gamma(2)$ | 4 | $\{1, \wp_{11}, \wp_{12}, \wp_{22}\}$ $\quad \partial \odot \underline{3} = \underline{4} \oplus \underline{2}$ |
| $\Gamma(3)$ | 9 | $\{1, \wp_{11}, \wp_{12}, \wp_{22},$ $\underline{4}$ $\wp_{111}, \wp_{112}, \wp_{122}, \wp_{222}, \Delta = \wp_{11}\wp_{22} - \wp_{12}^2\}$ |
| $\Gamma(4)$ | 16 | $\{1, \ldots, \Delta, \wp_{1111}, \ldots, \wp_{2222}, \partial_1\Delta, \partial_2\Delta\}$ |
| $\vdots$ | $\vdots$ | |
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# Further Reading

- C. Athorne.
  *Identities for hyperelliptic $\wp$-functions of genus one, two and three in covariant form.*
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- M. England.
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## Further Information

Matthew.England@glasgow.ac.uk
Christopher.Athorne@glasgow.ac.uk