On polynomial integrals of a natural mechanical system on a two-dimensional torus

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Let us consider Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p_1},\ \dot{y} = \frac{\partial H}{\partial p_2},\ \dot{p_1} = -\frac{\partial H}{\partial x},\ \dot{p_2} = -\frac{\partial H}{\partial y}$$

with Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + V(x,y),\ V((x,y)+\lambda) = V(x,y), \lambda \in \Lambda$$

where $\Lambda \in \mathbb{R}^2$ is a lattice of periods.

For the first integral F we have

$$\{H,F\} = \left(\frac{\partial H}{\partial p_1}\frac{\partial F}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial F}{\partial p_1}\right) + \left(\frac{\partial H}{\partial p_2}\frac{\partial F}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial F}{\partial p_2}\right) = 0.$$

There are two well known integrable cases:

1) $V(x,y) = V(\alpha x + \beta y)$, $(\alpha, \beta) \in \Lambda^*$,

$$F_1 = \alpha p_2 - \beta p_1.$$

2) $V(x,y) = V_1(\alpha_1 x + \beta_1 y) + V_2(\alpha_2 x + \beta_2 y)$, $(\alpha_i, \beta_i) \in \Lambda^*$,

$$F_2 = (d_1 + d_2)p_1^2 + 4p_1p_2 - (d_1 + d_2)p_2^2 + 2(d_1 - d_2)(V_1 + V_2),$$

$d_i = \alpha_i/\beta_i$.

Is there potential V with polynomial integral F which is not reducible to the case 1 or 2?

$$F = F_n + F_{n-1} + \cdots + F_0,$$

F_k is a homogenous polynomial in p_1 and p_2 of degree k

$$F_k = \sum_{j=0}^{k} b_j^k(x,y) p_1^{k-j} p_2^j.$$

We can assume that

$$F = F_n + F_{n-2} + F_{n-4} + \dots,$$

Indeed

$$\{H, F_k\} = G_{k+1} + G_{k-1}.$$

Maupertuis principle. Hamiltonian system

$$H' = \frac{p_1^2 + p_2^2}{2(h - V)}, \ h > \max V,$$

is integrable

$$F' = F_n + F_{n-2}\left(\frac{p_1^2 + p_2^2}{2(h - V)}\right) + F_{n-4}\left(\frac{p_1^2 + p_2^2}{2(h - V)}\right)^2 + \dots$$

Let us consider the case $n = 3$:

$$F_3 = a_0 p_1^3 + a_1 p_1^2 p_2 + a_2 p_1 p_2^2 + a_3 p_2^3 + b_0 p_1 + b_1 p_2, \ a_j = a_j(x, y), \ b_j = b_j(x, y).$$

$$\{H, F_3\} = G_4 + G_2 + G_0,$$

The condition $G_4 = 0$ is equivalent to

$$a_{0_x} = 0, \ a_{0_y} + a_{1_x} = 0,$$

$$a_{1_y} + a_{2_x} = 0, \ a_{2_y} + a_{3_x} = 0, \ a_{3_y} = 0.$$

Consequently, $a_j = const.$

$G_2 = 0$ is equivalent to

$$3a_0 V_x + a_1 V_y - b_{0_x} = 0,$$

$$2a_1 V_x + 2a_1 V_y - b_{0_y} - b_{1_x} = 0,$$

$$a_2 V_x + 3a_3 V_y - b_{1_y} = 0.$$

$G_4 = 0$ is equivalent to

$$b_0 V_x + b_1 V_y = 0.$$

Theorem *(Misha Bialy, 1985) If b_0, b_1, V is a periodic solution of the system with the lattice of periods $\Lambda = \mathbb{Z}^2$*

$$b_j(x+1, y) = b_j(x, y+1) = b_j(x, y),\ V(x+1, y) = V(x, y+1) = V(x, y)$$

then $V = V(ax + by)$.

Geodesic flow. $H' = \frac{p_1^2+p_2^2}{2V}$,

$$F' = d_0 p_1^3 + d_1 p_1^2 p_2 + d_2 p_1 p_2^2 + d_3 p_2^3,\ \ d_j = d_j(x,y)$$

Theorem *(V.N. Kolokoltsov)*

$$d_2 = d_0 + c_1,\ \ d_3 = d_1 + c_2.$$

$$F' = a_0 p_1^3 + a_1 p_1^2 p_2 + a_2 p_1 p_2^2 + a_3 p_2^3 + b_0 H' p_1 + b_1 H' p_2, \quad b_j = b_j(x,y).$$

$$d_0 = \frac{b_0 + 2a_0 V}{2V},\ \ d_1 = \frac{b_1 + 2a_1 V}{2V},$$

$$c_1 = a_2 - a_0,\ \ c_2 = a_3 - a_1.$$

Let

$$a_1 = -3a_3,\ \ a_2 = -3a_0.$$

The condition

$$\{H', F'\} = 0$$

is equivalent

$$(b_0V)_x + (b_1V)_y = 0,$$

$$b_{0_y} + b_{1_x} - 6a_0V_y - 6a_3V_x = 0,$$

$$b_{1_y} - b_{0_x} - 6a_0V_x + 6a_3V_y = 0.$$

Let us consider the case $n = 4$

$$F_4 = a_0 p_1^4 + a_1 p_1^3 p_2 + a_2 p_1^2 p_2^2 + a_3 p_1 p_2^3 + a_4 p_2^4 + b_0 p_1^2 + b_1 p_1 p_2 + b_2 p_2^2 + q.$$

$$\{H, F_4\} = G_5 + G_3 + G_1,$$

The condition $G_5 = 0$ is equivalent to

$$a_{0_x} = 0,\ a_{0_y} + a_{1_x} = 0,\ a_{1_y} + a_{2_x} = 0$$

$$a_{2_y} + a_{3_x} = 0,\ a_{3_y} + a_{4_x} = 0\ a_{4_y} = 0.$$

Consequently, $a_j = const.$

The condition $G_3 = 0$ is equivalent to

$$4a_0 V_x + a_1 V_y - b_{0_x} = 0, \tag{1}$$

$$3a_1 V_x + 2a_2 V_y - b_{0_y} - b_{1_x} = 0, \tag{2}$$

$$2a_2 V_x + 3a_3 V_y - b_{1_y} - b_{2_x} = 0,$$

$$a_3 V_x + 4a_4 V_y - b_{2_y} = 0.$$

From (1) and (2)

$$4a_0 V_{xy} - 3a_1 V_{xx} - 2a_2 V_{xy} + a_1 V_{yy} + b_{1_{xx}} = 0.$$

We have

$$a_3 V_{xxxx} - 2(a_2 - 2a_4) V_{xxxy} + 3(a_1 - a_3) V_{xxyy} - 2(2a_0 - a_2) V_{xyyy} - a_1 V_{yyyy} = 0,$$

$$V = \sum v_{mn} e^{2\pi i m x + n y}.$$

If $v_{mn} \neq 0$ then m/n satisfies the equation

$$a_3 x^4 - 2(a_2 - 2a_4) x^3 + 3(a_1 - a_3) x^2 - 2(2a_0 - a_2) x - a_1 = 0.$$

We have

$$V = V_1(\alpha_1 x + \beta_1 y) + V_2(\alpha_2 x + \beta_2 y) + V_3(\alpha_3 x + \beta_3 y) + V_4(\alpha_4 x + \beta_4 y).$$

Theorem *(Misha Bialy, 1985) If there is a polynomial integral of the fourth degree ($\Lambda = \mathbb{Z}^2$) and*

$$V \neq V_1(x) + V_2(x+y) + V_3(y) + V_4(x-y)$$

then there is an integral of the first degree or second degree.

If

$$V = \frac{a + b \sin x}{\cos^2 x} + \cos(x+y) + \frac{c + d \sin y}{\cos^2 y} - \cos(y-x),$$

$a, b, c, d \in \mathbb{R}$, then there is the integral

$$F = \frac{p_1^2 p_2^2}{2} + \frac{c + d \sin y}{\cos^2 y} p_1^2 + (\cos(y+x) + \cos(y-x)) p_1 p_2 + \frac{a + b \sin x}{\cos^2 x} p_2^2 + f.$$

Theorem *(S. Agapov, D. Aleksndrov) If*

$$V = V_1(x) + V_2(x+y) + V_3(y) + V_4(x-y)$$

is analytic and there is an integral of the fourth degree then there is an integral of the first degree or second degree.

Let

$$V_i(z) = \sum_{n=-\infty}^{+\infty} v_i^n e^{2\pi i n z}.$$

From $V_i \in \mathbb{R}$ it follows $v_i^n = \overline{v_i^{-n}}$.

From the condition $\{H, F\} = 0$ we get

Lemma 1

$$(m+n)(mv_2^n v_3^{m-n} - nv_1^{-m+n} v_2^m) - (m-n)(nv_1^{m+n} v_4^m + mv_3^{m+n} v_4^{-n}) = 0,$$

where $m, n \in \mathbf{Z}$.

Lemma 2

$$|v_1^n| = |v_3^n|, \ |v_2^n| = |v_4^n|.$$

Let $u_i^k = \frac{v_i^k}{k}$, then

$$u_2^n u_3^{m-n} + u_1^{-m+n} u_2^m - u_1^{m+n} u_4^m + u_3^{m+n} u_4^{-n} = 0.$$

$$\begin{cases} u_2^n u_3^{m-n} + u_1^{-m+n} u_2^m - u_1^{m+n} u_4^m + u_3^{m+n} u_4^{-n} = 0 & (m,n) \\ u_2^n u_3^{-(m+n)} + u_1^{m+n} u_2^{-m} - u_1^{-m+n} u_4^{-m} + u_3^{-m+n} u_4^{-n} = 0 & (-m,n) \\ u_2^m u_3^{-m+n} + u_1^{m-n} u_2^n - u_1^{m+n} u_4^n + u_3^{m+n} u_4^{-m} = 0 & (n,m) \\ u_2^m u_3^{-(m+n)} + u_1^{m+n} u_2^{-n} - u_1^{m-n} u_4^{-n} + u_3^{m-n} u_4^{-m} = 0. & (-n,m) \end{cases}$$

Let

$$u_k^j = a_k^j + ib_k^j,$$

where $a_k^j, b_k^j \in \mathbb{R}$, then

The system is equivalent to

$$Aq = 0,$$

where

$$A = \begin{pmatrix} -a_4^m & b_4^m & -a_2^m & -b_2^m & -a_4^n & -b_4^n & a_2^n & -b_2^n \\ -b_4^m & -a_4^m & -b_2^m & a_2^m & b_4^n & -a_4^n & b_2^n & a_2^n \\ -a_2^m & -b_2^m & -a_4^m & b_4^m & -a_2^n & -b_2^n & a_4^n & -b_4^n \\ b_2^m & -a_2^m & b_4^m & a_4^m & -b_2^n & a_2^n & -b_4^n & -a_4^n \\ -a_4^n & b_4^n & a_2^n & -b_2^n & -a_4^m & -b_4^m & -a_2^m & -b_2^m \\ -b_4^n & -a_4^n & b_2^n & a_2^n & b_4^m & -a_4^m & -b_2^m & a_2^m \\ -a_2^n & -b_2^n & a_4^n & b_4^n & -a_2^m & -b_2^m & -a_4^m & -b_4^m \\ b_2^n & -a_2^n & -b_4^n & a_4^n & -b_2^m & a_2^m & b_4^m & -a_4^m \end{pmatrix},$$

$$q = (a_1^{m+n}, b_1^{m+n}, a_1^{m-n}, b_1^{m-n}, a_3^{m+n}, b_3^{m+n}, a_3^{m-n}, b_3^{m-n})^\top.$$

In the case $q \neq 0$ we have

$$\det A = 0 \Leftrightarrow |v_2^m| = |v_4^m|.$$

Theorem *(V.V. Kozlov, D.V. Treshsev) If V is trigonometric polynomial and there is an integral of the n-th degree then there is an integral of the first degree or fourth degree.*

Let us consider the general case

$$F = F_n + F_{n-2} + \ldots$$

$$F_n = a_0 p_1^n + a_1 p_1^{n-1} p_2 + \cdots + a_n p_2^n,$$

$$F_{n-2} = b_0 p_1^{n-2} + b_1 p_1^{n-3} p_2 + \cdots + b_{n-2} p_2^{n-2}.$$

$$\{H, F\} = G_{n+1} + G_{n-1} + \ldots$$

From the condition $G_{n+1} = 0$ we have

$$a_{0_x} = 0,\ a_{0_y} + a_{1_x} = 0,\ a_{1_y} + a_{2_x} = 0, \ldots, a_{n_y} = 0,$$

i.e. $a_j = const.$

The condition $G_{n-1} = 0$ is equivalent to

$$na_0 V_x + a_1 V_y - b_{0_x} = 0,$$

$$(n-1)a_1 V_x + 2a_2 V_y - b_{0_y} - b_{1_x} = 0,$$

$$(n-2)a_2 V_x + 3a_3 V_y - b_{1_y} - b_{2_x} = 0,$$

$$\cdots\cdots$$

$$a_{n-1} V_x + na_n V_y - b_{n-2_y} = 0.$$

From these equations we obtain

$$a_{n-1}\partial_x^n V + (na_n - 2a_{n-2})\partial_x^{n-1}\partial_y V -$$

$$-((n-1)a_{n-1} - 3a_{n-3})\partial_x^{n-2}\partial_y^2 V + \cdots \pm a_1 \partial_y^n V = 0,$$

$$V = \sum_{\lambda^* \in \Lambda^*} v_{\lambda^*} e^{2\pi i(\lambda_1^* x + \lambda_2^* y)}, \ \ \lambda^* = (\lambda_1^*, \lambda_2^*)$$

$$\Lambda^* = \{\lambda^* : (\lambda, \lambda^*) \in \mathbb{Z}, \lambda \in \Lambda\}.$$

If $v_{\lambda^*} \neq 0$ then λ_1^*/λ_2^* satisfies the equation

$$a_{n-1}x^n + (na_n - 2a_{n-2})x^{n-1} - ((n-1)a_{n-1} - 3a_{n-3})x^{n-2} + \cdots \pm a_1 = 0.$$

$$V = V_1(\alpha_1 x + \beta_1 y) + \cdots + V_n(\alpha_n x + \beta_n y).$$

Kozlov and Denisova considered the equations

$$G_{n+1} = G_{n-1} = G_{n-3} = 0:$$

Let us assume that

$$V = V_1(y) + V_2(\alpha_2 x + \beta_2 y) + \cdots + V_n(\alpha_n x + \beta_n y),$$

i.e. $a_1 = 0$. Then

Theorem *(N.V. Denisova, V.V. Kozlov)*

$$a_3 = 0.$$

Theorem *(N.V. Denisova, V.V. Kozlov) If there is a polynomial integral of the third degree (Λ is arbitrary) then there is an integral of the first degree.*

Theorem *(N.V. Denisova, V.V. Kozlov) If there is a polynomial integral of the fourth degree (Λ is arbitrary) and*

$$V \neq V_1(x) + V_2(x+y) + V_3(y) + V_4(x-y)$$

then there is an integral of the first degree or second degree.

Let us assume that there is an integral of odd degree in momenta

$$F = F_{2n+1} + F_{2n-1} + \dots$$

Let us introduce the following notation:

$$\alpha = (2n+1)!!a_0 + (2n-1)!!1!!a_2 + (2n-3)!!3!!a_4 + \dots + 1!!(2n-1)!!a_{2n},$$

$$\beta = (2n+1)!!a_{2n+1} + (2n-1)!!1!!a_{2n-1} + \dots + 1!!(2n-1)!!a_1,$$

where $(2k+1)!! = 1 \cdot 3 \cdots (2k+1)$.

Theorem *(M.) If V is analytical function then either $\alpha = \beta = 0$ or there is an integral of the first degree in momenta.*

$$\mathcal{F}_y V_x - \mathcal{F}_x V_y + \alpha V_x + \beta V_y = 0,$$

$$\mathcal{F} = G(V) + \beta x - \alpha y.$$

$$3a_0V_x + a_1V_y - b_{0_x} = 0, \tag{1}$$

$$2a_1V_x + 2a_1V_y - b_{0_y} - b_{1_x} = 0,$$

$$a_2V_x + 3a_3V_y - b_{1_y} = 0, \tag{2}$$

$$b_0V_x + b_1V_y = 0.$$

Let

$$\mathcal{F} = -\frac{b_1}{V_x} = \frac{b_0}{V_y}$$

or equivalent

$$b_0 = \mathcal{F}V_y,\ \ b_1 = -\mathcal{F}V_x.$$

From (1) and (2) we have

$$b_{0_x} + b_{1_y} = (3a_0 + a_2)V_x + (a_1 + 3a_3)V_y.$$

From hire we obtain

$$\mathcal{F}_y V_x - \mathcal{F}_x V_y + (3a_0 + a_2)V_x + (a_1 + 3a_3)V_y = 0.$$

In the case $n = 5$ we have

$$a_1 + a_3 + a_5 = 0,\ 5a_0 + a_2 + a_4 = 0.$$

Theorem *(M.) Let the potential V be periodic in x and y,*

$$V(x+1, y) = V(x, y+1) = V(x, y).$$

Further assume that there is an integral of the fifth degree in momenta. Then there exists an integral of the first degree in momenta.

A.Mironov. On polynomial integrals of a mechanical system on a two-dimensional torus. Izvestiya: Mathematics. 2010. Vol. 74. N. 4. P. 805-817.